Nagata and the Mori Cone Perspective

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Rays

- I will consider X_n the plane blown–up at n *very general* points.
- A divisor class on X_n can be denoted by $(d; h_1, \ldots, h_n)$.
- Ray in $\mathrm{Pic}_{\mathbb{R}}$: all non-negative multiples of a vector
- Rational ray in $\mathrm{Pic}_{\mathbb{R}}$: all non-negative multiples of a rational vector in $\mathrm{Pic}_{\mathbb{Q}}$
- (same as all non-negative multiples of an integer vector in Pic)
- Example: The Nagata Ray, generated by $(\sqrt{n}; 1, 1, \ldots, 1)$.
- Example: The Anticanonical Ray, generated by $-K = (3; 1, 1, \ldots, 1)$.
- Example: A (-1)-Ray, generated by the class E of a (-1)-curve.
- Note: If R is a ray, then neither $\deg(R)$ nor R^2 makes sense, but their sign makes sense.

Effective classes

- An integer class is effective if there exists a curve.
- A rational class in $\mathrm{Pic}_{\mathbb{Q}}$ is effective if some integer multiple is effective.
- A rational ray is effective if it can be generated by an effective rational class.
- The sum of two effective classes is effective, so effective classes form a cone.
- Riemann-Roch: If R is rational, $\deg(R) > 0$ and $R^2 > 0$ then R is effective.

The Interesting Cones

- The *Effective Cone*: the cone Eff in $\mathrm{Pic}_{\mathbb{R}} = \mathrm{Pic} \otimes \mathbb{R}$ generated by the effective rays.
  This is the cone we really want to understand.
- The *Mori Cone*: the closure $\mathrm{Mori} = \overline{\mathrm{Eff}}$ in $\mathrm{Pic}_{\mathbb{R}}$ of the Effective Cone.
- A class/ray C is *nef* if $C \cdot D \geq 0$ for all effective D.
- The *Nef Cone*: the cone Nef of nef classes, the dual of the Mori Cone.
- Mori and Nef cones are closed; Effective cone may not be.
- The *Nonnegative Cone*: the cone Q of classes C such that $\deg(C) \geq 0$ and $C^2 \geq 0$.
- The *Ample Cone*: the interior of the Nef Cone, intersected with the positive self-intersection cone.

The Cone Theorem

- Let $K^{\geq} = \{C \mid C \cdot K \geq 0\}$ and $K^{\leq} = \{C \mid C \cdot K \leq 0\}$
- Define $\mathrm{Mori}^{\geq} = \mathrm{Mori} \cap K^{\geq}$ and $\mathrm{Mori}^{\leq} = \mathrm{Mori} \cap K^{\leq}$
- Clearly $\mathrm{Mori} = \mathrm{Mori}^{\geq} + \mathrm{Mori}^{\leq}$ as cones.
- Note that if E is a (-1)-curve, then $E \in \mathrm{Mori}^{\leq}$.

Theorem (Mori's Cone Theorem)

Mori $= \mathrm{Mori}^{\geq} + \sum_{\alpha} E_{\alpha}$ *where each* E_{α} *is a* (-1)*-curve. Moreover the* E_{α}*'s can only accumulate on the boundary where the intersection with* K *is zero.*

The cases $n \leq 9$

- If $n \leq 8$, then $-K$ is ample, so Eff $\subset K^{<}$, so Mori $\subset K^{\leq}$.
- I.e., Mori$^{\geq}$ is empty.
- In this case (the Del Pezzo case), the Mori $=$ Eff and is rational polyhedral, finitely generated by a finite number of (-1)-curves.
- If $n = 9$, there are infinitely many (-1)-curves, but they accumulate only to $-K$.
- In this case Mori $= -K + \sum_\alpha E_\alpha$

$n \geq 10$: The Nagata Ray

- Recall D is *nef* if $D \cdot C \geq 0$ for all curves C.
- Nagata's Conjecture: If $n \geq 10$ and $(d, \underline{m})$ is effective, then $\sqrt{n}d - \sum_i m_i \geq 0$
- This exactly says that $N = (\sqrt{n}; 1, 1, \ldots, 1)$ is nef when $n \geq 10$
- Note that $N^2 = 0$.
- $N \cdot K = -3\sqrt{n} + n$ which is > 0 if $n \geq 10$.

Cones, Extremal Rays

- Cones: Effective (Eff), Mori, Nef, Non-Negative Q, with $\leq$, $\geq$ etc.
- Cones are unions of rays
- A ray is *extremal* for a cone if it is not in the sum of two other rays in the cone
- (Extremal: on the boundary, not in the interior of a (linear) face.)
- (-1)-Rays are extremal rays and Mori$^{<}$ is *locally polyhedral*.

The Nefness Lemma

A useful lemma (not hard to prove, but uses machinery) to probe the boundary of the cones:

Lemma

Suppose R is a ray, such that:

- *R is rational, $R^2 = 0$, and R is not effective*

Then:

- *R is nef and is on the boundary of the* Mori *and* Eff *cones*
- *R is extremal for both* Mori *and* Nef *cones*
- *$R \cdot K \geq 0$.*

(Proof involves Zariski decomposition argument.)

Good and Wonderful Rays

- Rational, not effective, $R^2 = 0$: "Good Ray"
- R is Good $\implies$ R is extremal for Mori and Nef cones, on boundary
- Difficult to prove R is Good: have to show primitive integral vector and all multiples are not effective, i.e. no polynomial exists.
- "Wonderful Ray": irrational, $R^2 = 0$, Nef.
- Wonderful ray: irrational limit of good rays. These are extremal also, for Mori and Nef.
- Wonderful rays would be evidence for non-polyhedral nature of Mori.

Conjectures

Conjecture (De Fernex)

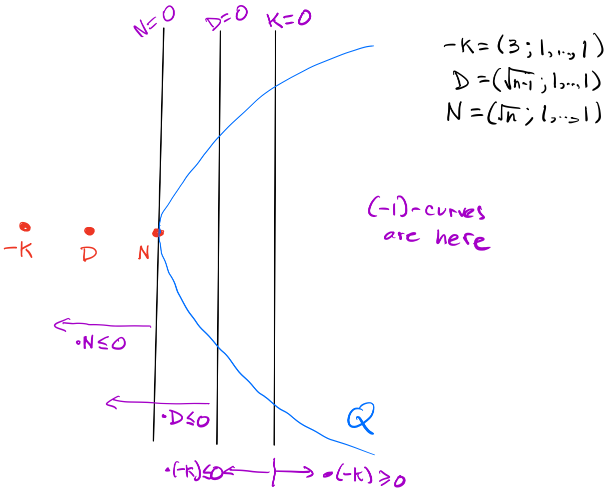
Mori $= Q^{\geq} + \sum_{\alpha} E_{\alpha}$

- The *de Fernex Ray* $D = (\sqrt{n-1}; 1, 1, \ldots, 1)$.

Conjecture (De Fernex, C.- Harbourne - Miranda - Roé)

$(n > 10)$ R rational, $R^2 = 0$ and $R \cdot D \leq 0 \implies R$ is not effective. Hence R is good, and nef.
If R is irrational, $R^2 = 0$ and $R \cdot D \leq 0$, then R is nef (hence wonderful).

- Note that this would imply the Nagata Conjecture: the Nagata Ray N in this range has $N \cdot D < 0$ and is irrational, hence would be nef.
- By the way: any good/wonderful ray produces a counterexample to Hilbert's 14-th problem.


N=0
D=0
K=0
−K = (3; 1,…,1)
D = (√(n−1); 1,…,1)
N = (√n; 1,…,1)
(−1)-curves are here
−K
D
N
•N≤0
•D≤0
•(−K)≤0
•(−K)≥0
Q

Examples of Good Rays

Good rays are difficult but not impossible to discover.

Theorem (C.-Harbourne-Miranda-Roé 2016)

For every $n \geq 10$, good rays exist.

- Proof was constructive.
- A degeneration argument is used to prove non-effectivity.

Example

$(n = 10)$: $(13; 5, 4^9)$ is good.

Ciliberto/Miranda constructed more families in 2021.

Wonderful Rays

No wonderful rays were known until 2021.

Theorem (C-Miranda-Roé 2021)

- *For all* $n \geq 10$, *wonderful rays* R *with* $R \cdot K = 0$ *exist.*
- *For all* $n \geq 13$, *wonderful rays* R *with* $R \cdot K > 0$ *exist.*
- *For all* $n = 14$, *or* $n \geq 13$ *such that* $n - 4$ *is a square, or* $n \geq 18$ *such that* $n - 2$ *is a square, wonderful rays* R *with* $R \cdot D < 0$ *exist.*

Example ($n = 13$)

The ray spanned by $d = 1428$ and
$m_1 = \cdots = m_4 = 21(15 + \beta)$;
$m_5 = \cdots = m_{11} = (462 - 5\beta)$;
$m_{12} = m_{13} = 14(15 + \beta)$
with $\beta = \sqrt{21}$ is wonderful.

Outline of the proof

- Proof: starts with a good ray R_0.
- Uses a Cremona transformation A to transform R_0; A is represented by a matrix acting on the parameters d, m_i.
- $R_k = A^k R_0$ is good for all k, and converges to an eigenvector ray R_∞ for A which is irrational. Such an R_∞ is always perpendicular to K.
- To obtain DeFernex-negative rays, or K-positive rays, we use a degeneration of four points colliding to one.
- This produces systems T_k colliding to $R_k = A^k R_0$.
- Since the resulting collided system R_k is not effective, the 'uncollided' system T_k is not effective by semi-continuity.
- The T_k systems converge to the desired wonderful rays.

A bit more evidence

- For $n = 13$, we have found a one-dimensional arc at the boundary of the Mori cone, consisting of an infinite set of good rays that are dense in that arc.
- This arc lies in $Q = 0$, and in the deFernex-negative range.
- In 2023, we have found entire 'belts' of regions that are at the boundary of the Mori cone, lying inside $Q = 0$, providing further evidence for the Delta-Conjecture.
- These 'belts' are 8-dimensional.

Thank you!