Math 635, USC Spring 2026

Cluster varieties: algebra, topology, geometry, duality

Lecture 1

1/12/26

Roughly speaking:

- a <u>cluster variety</u> is a complex algebraic variety obtained by gluing together many copies of $(\mathbb{C}^*)^n$, where the gluing maps take a very particular form,
- a <u>cluster algebra</u> is the algebra of regular functions $f: V \longrightarrow \mathbb{C}$ on a cluster variety

Fomin–Zelevinsky, early '00s: introduced cluster algebras

Arise in many parts of math and physics as kind of "universal model" for mutation/wall-crossing phenomena:

- quiver representation theory
- ~~Teichmull~~ Teichmüller theory
- Poisson geometry
- Grassmannians
- total positivity
- QFT scattering amplitudes (~~amplitudre~~ amplituhedron)
- integrable systems
- string theory (BPS states)

etc

<u>Gross–Hacking–Keel–Kontsevich '18</u>:

- constructed canonical bases for cluster algebras
- established ~~positivity of the~~ Positive Laurent phenomenon
- proof uses mirror symmetry for <u>log Calabi–Yau varieties</u>

can think of as generalization of toric varieties (related to almost toric fibrations in symplectic geometry)

- many strong applications in representation theory, e.g. canonical bases for finite-dimensional irreducible representations of $SL_n(\mathbb{C})$

originally found independently by Lusztig and Kashiwara in early '90s using quantum groups

amazingly, the construction of GHKK uses only general geometry – no rep. thry!

Total positivity

Def: $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is totally positive (TP) is all of its minors are positive.

Gantmacher–Krein '30's: A TP $\Rightarrow$ eigenvalues are real, positive, and distinct

Binet–Cauchy theorem: The TP matrices in $G = SL_n(\mathbb{C})$ are closed under multiplication, and hence form a multiplicative semigroup $G_{>0}$.

Lusztig: Extended definition of $G_{>0}$ for other semisimple Lie groups G.

More generally: If a given complex algebraic variety Z has a distinguished family Δ of regular functions $Z \to \mathbb{C}$, we define the TP variety by

$$Z_{>0} := \{ z \in Z \mid f(z) > 0 \ \forall f \in \Delta \}$$

Ex: For $Z = \mathrm{Mat}_{n\times n}(\mathbb{C}), GL_n(\mathbb{C}), SL_n(\mathbb{C})$, $\Delta =$ minors, recover above notion of TP

Ex: Grassmannian $Gr_{k,m}(\mathbb{C}) = \{k\text{-dim'l linear subspaces of } \mathbb{C}^m\}$

$\Delta =$ Plücker coordinates ← slight scaling ambiguity

Ex: partial flag manifolds, homogeneous spaces for semisimple complex Lie groups, etc

Lemma: $A \in \mathrm{Mat}_{n\times n}$ has $\binom{2n}{n} - 1$ minors

pf: $\# = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k}$

Vandermonde's identity: $\binom{m+w}{r} = \sum_{k=0}^{n} \binom{m}{k}\binom{w}{r-k}$

Setting $m=w=r=n \rightsquigarrow \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}$

(both sides count: given committee with m men, w women, how many subcommittees with r members?)

Q: Can we check that $A \in \mathrm{Mat}_{n\times n}$ is TP by only testing a subset of the $\binom{2n}{n} - 1$ minors? How many tests are needed?

← i.e. want "efficient TP testing"

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}$

$\delta := ad - bc \Rightarrow d = \dfrac{\delta + bc}{a}$.

So if $a, b, c, \delta > 0$, so is d.

Reduce $\binom{4}{2} - 1 = 5$ checks to 4 checks.

Plücker coordinates on Grassmannians:

Given $A \in \mathrm{Mat}_{k \times m}$ of rank k $\rightsquigarrow$ row span $[A] \in Gr_{k,m}$

For $J \subset \{1,\dots,m\}$, $|J|=k$ $\rightsquigarrow$ Plücker coordinates $P_J(A) :=$ $k\times k$ minor of A corresponding to J

Note: For $A, B \in \mathrm{Mat}_{k\times m}$ with $[A]=[B]$ (i.e. same row spans) $(P_J(A))_{|J|=k}$ and $(P_J(B))_{|J|=k}$ agree up to common rescaling, i.e. get

$$Gr_{k,m} \longrightarrow \mathbb{CP}^N \quad \text{for } N = \binom{m}{k}-1.$$

In fact this is an embedding, the <u>Plücker embedding</u>.

Let $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ = coord. ring of $\mathrm{Mat}_{k\times m}$, i.e. the polynomial algebra in variables x_{ij} for $1 \le i \le k$, $1 \le j \le m$

Def: The Plücker ring $R_{k,m}$ is the subring of $\mathbb{C}[\mathrm{Mat}_{k\times m}]$ generated by P_J over $J \in \{1,\dots,m\}$, $|J|=k$.

Claim: the ideal of relations in $R_{k,m}$ is gen'd by certain quadratic relations called the Grassmann–Plücker relations.

Def: The totally positive Grassmannian $Gr^+_{k,m}$ is the subset of $Gr_{k,m}$ of those pts whose Plücker coords are all positive (up to common scaling).

Note: For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, $[A] \in Gr^+_{k,m}$ iff all $k\times k$ minors of A have the same sign.

positive WLOG

Q: For $A \in \mathrm{Mat}_{k\times m}(\mathbb{R})$, can we verify that all $k\times k$ minors are positive by only checking a subset of the $\binom{m}{k}$ minors? How many tests are needed?

Positivity testing for $Gr_{2,m}$

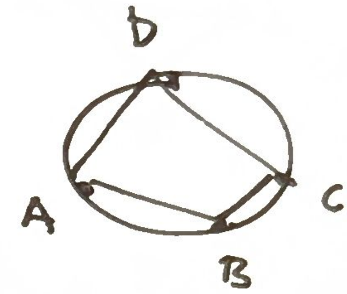
Claim: Given $A \in Mat_{2\times m}$, put $P_{ij} := P_{\{i,j\}}$ for $1 \le i < j \le m$. To check that all 2×2 minors $P_{ij}(A) > 0$, suffices to check only $2m-3$ special ones.

Note: $2m-3 = \dim Gr_{2,m} + 1$

Lemma: For $1 \le i < j < k < l \le m$, have three-term Grassmann–Plücker relations:

$$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk}$$

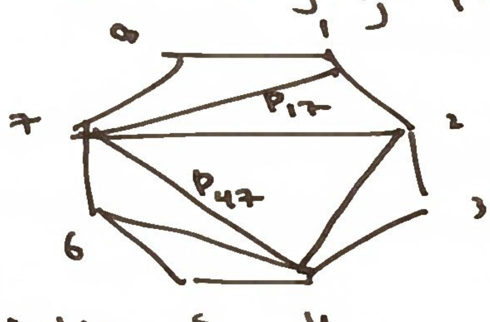
[Rmk: For inscribed quadrilateral Ptolemy's thm (2nd century) gives
$AC \cdot BD = AB \cdot CD + BC \cdot AD$]



Ex: $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$ wts $P_{13}P_{24} = P_{12}P_{34} + P_{14}P_{23}$, i.e.

$$(ag-ce)(bh-df) = (af-be)(ch-dg) + (ah-de)(bg-cf) \checkmark$$

Put P_m = regular m-gon, T = triangulation.



To each side or diagonal associate P_{ij}, where i,j are the end pts

Cluster variables: P_{ij} ranging over diagonals

frozen variables: P_{ij} ranging over sides

extended cluster: $\{\text{cluster vars}\} \cup \{\text{frozen vars}\} =: \tilde{x}(T)$

Note: extended cluster has $2m-3$ vars, and we claim that these are algebraically independent.

Ex: In above picture, have cluster variables $P_{17}, P_{27}, P_{47}, P_{46}, P_{24}$
frozen variables $P_{12}, P_{23}, \dots, P_{78}, P_{18}$

Thm Each P_{ij} for $1 \le i < j \le n$ can be written as a subtraction-free rational expression in the elements of a given extended cluster $\tilde{x}(T)$.

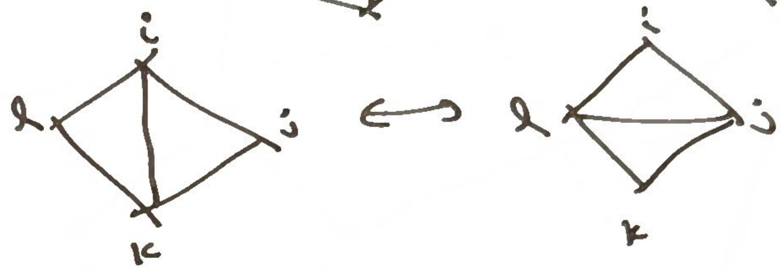
Cor: ~~In particular~~ ~~If~~ each ~~variable in~~ $\underbrace{P_{ij} \in \tilde{x}(T)}_{2m-3 \text{ of these}}$ evaluates positively on given $A \in Mat_{2\times m}$, then $\underbrace{\text{all of the 2x2 minors of } A}_{\binom{m}{2} \text{ of these}}$ are positive.

Pf of thm: Follows by combining

(1) each p_{ij} appears as an elt of an extended cluster $\tilde{x}(T)$ for some triangulation T of $\mathbb{P}_m$

(2) any two triangulations of $\mathbb{P}_m$ are related by a sequence of flips

(3) For a flip



,

replace p_{ik} with p_{lj}.

Using three-term GP relation, have $p_{ik} = \dfrac{p_{ij}p_{lk} + p_{il}p_{jk}}{p_{jl}}$

Rmk: In fact, each Plücker coordinate p_{ij} can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates from $\tilde{x}(T)$.

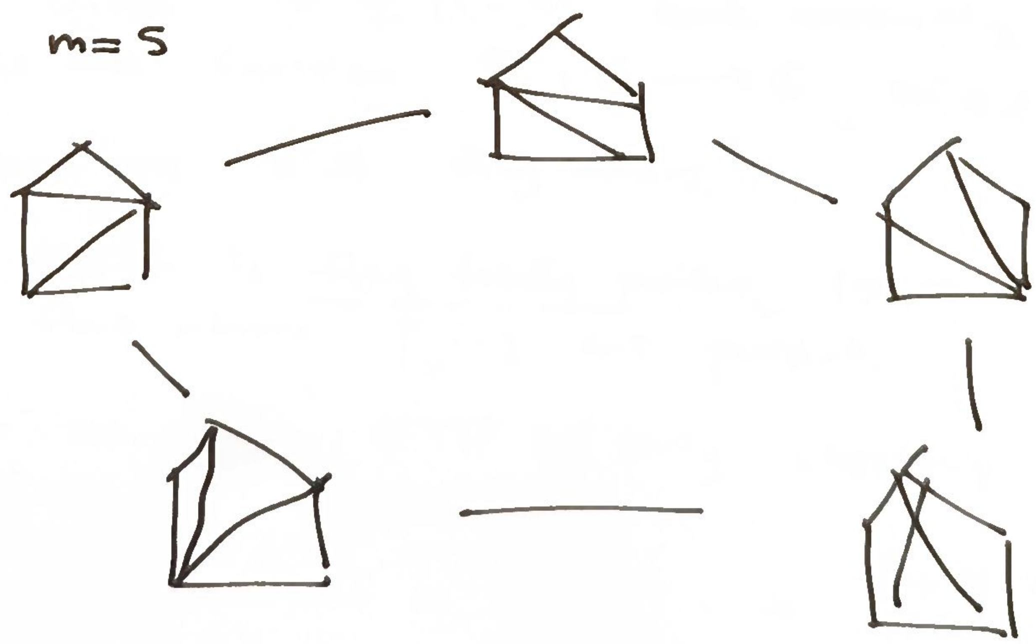
Example of ~~positivity~~ positive Laurent phenomenon.

Combinatorics of flips encoded by graph:

- vertices are ~~tri~~ triangulations
- edges are flips

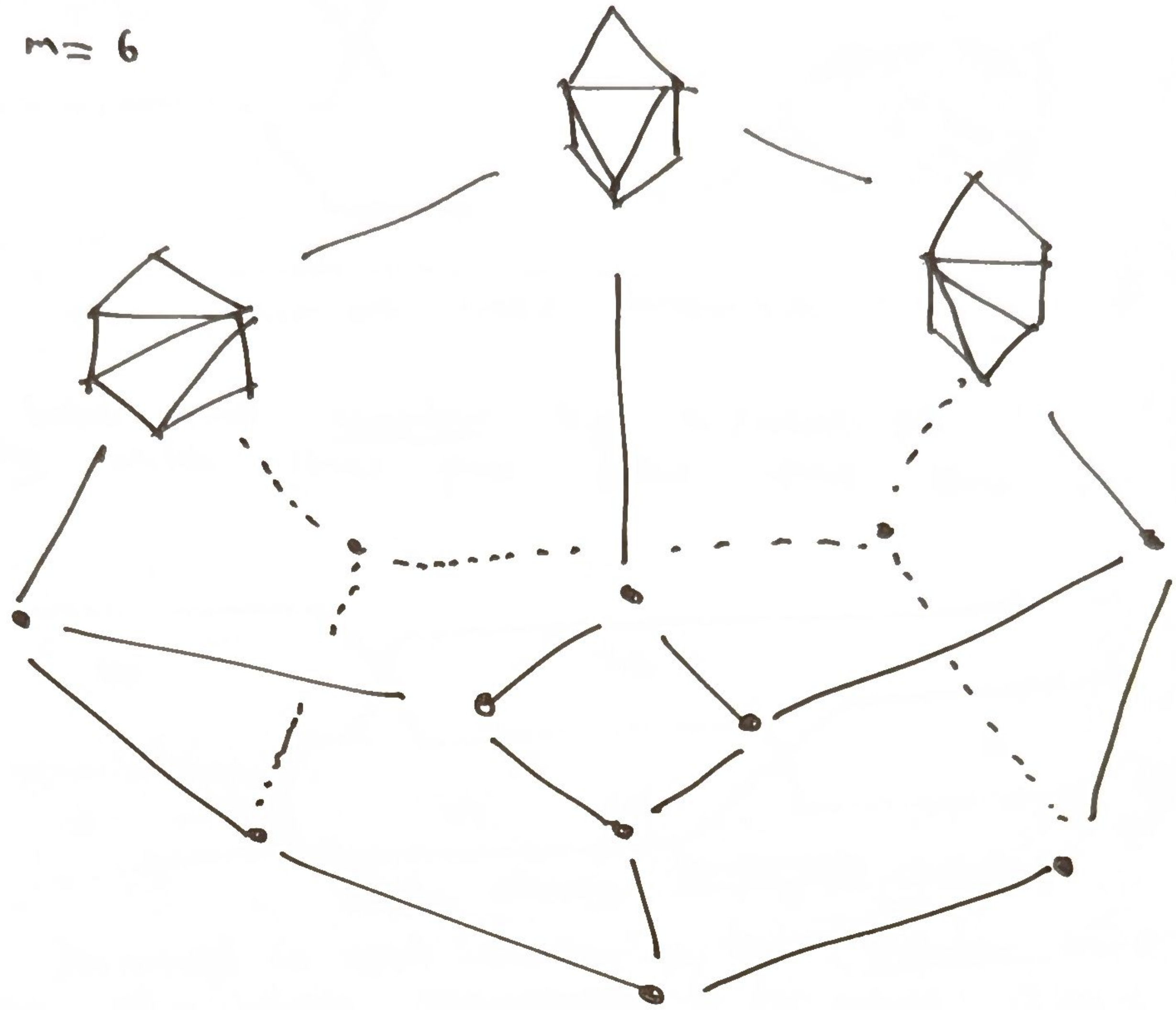
Each vertex has degree $m-3$. In fact, this is the 1-skeleton of an $(m-3)$-diml convex polytope called the associahedron (discovered by Stasheff).

Ex: $m=5$



Wiring diagrams:

Ex: $n = 6$



Def: A cluster monomial is a monomial in the variables of a given extended cluster $\tilde{x}(T)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{2,n}$.

Lecture 2

1/14/26

Before moving to TP for $n \times n$ matrices, we discuss an intermediate notion called "flag positivity". Put $G = SL_n$.

Def. Given $J \subsetneq \{1, \dots, n\}$ ~~none~~ nonempty, the flag minor P_J is the function $P_J : G \longrightarrow \mathbb{C}$, $z = (z_{ij}) \mapsto \det\left(z_{ij} \,\middle|\, \begin{smallmatrix} i \le |J| \\ j \in J \end{smallmatrix}\right)$

$|J| \times |J|$ minor which is "top-justified"

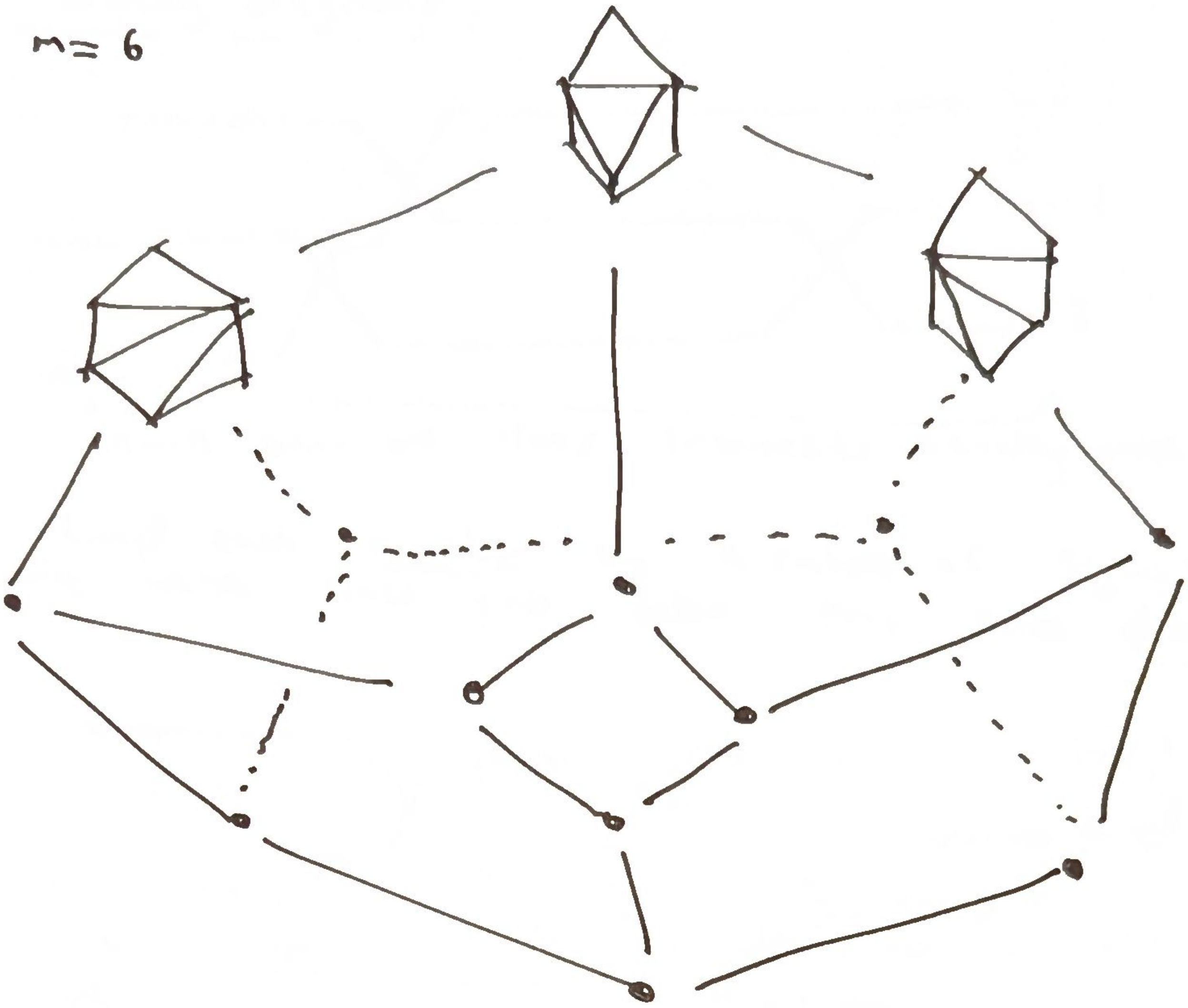
Note: there are $2^n - 2$ flag minors.

Def: $z \in G$ is flag totally positive (FTP) is all flag minors $P_J(z)$ are positive.

Q.: Can we check FTP by only checking a subset of the $2^n - 2$ flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.

Ex: $n=6$



Def: A <u>cluster monomial</u> is a monomial in the variables of a given extended cluster $\tilde{x}(T)$.

Thm (19th century invariant theory): The set of all cluster monomials give a linear basis for the Plücker ring $R_{2,m}$.

Lecture 2

1/14/26

Before moving to TP for $n\times n$ matrices, we discuss an intermediate notion called "flag positivity". Put $G = SL_n$.

Def. Given $J \subsetneq \{1,\dots,n\}$ ~~none~~ nonempty, P_J is the function $P_J : G \longrightarrow \mathbb{C}$, $z=(z_{ij}) \mapsto \det(\underbrace{z_{ij} \mid \begin{smallmatrix} i \le |J| \\ j\in J\end{smallmatrix}}_{|J|\times|J| \text{ minor which is "top-justified"}})$, the <u>flag minor</u>

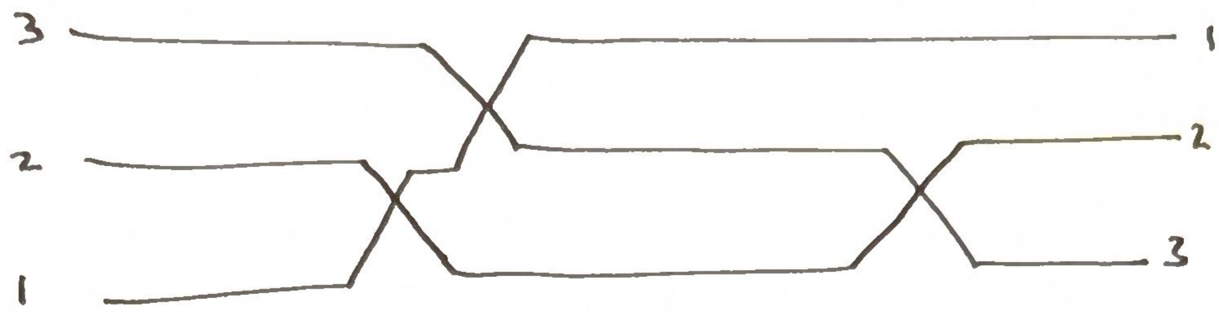
Note: there are 2^n-2 flag minors.

Def: $z \in G$ is <u>flag totally positive</u> (FTP) is all flag minors $P_J(z)$ are positive.

Q: Can we check FTP by only checking a subset of the 2^n-2 flag minors.

Claim: It suffices to check only $\frac{(n-1)(n+2)}{2}$ special flag minors.

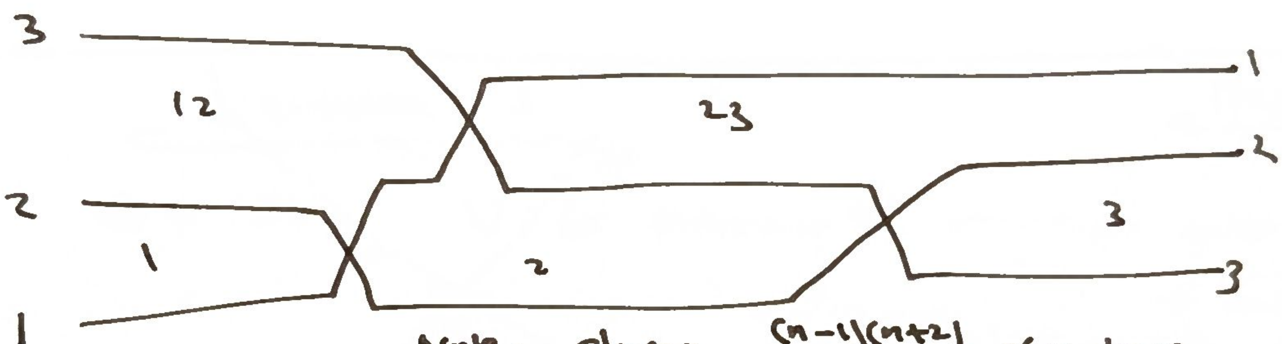
Wiring diagrams:



(case $n=3$)

each pair of lines intersects exactly once

We label each chamber by a subset of $\{1,\dots,n\}$ indicating which lines pass below that ~~?~~ chamber



Note: always $\frac{(n-1)(n+2)}{2}$ chambers

Associated to each chamber is its chamber minor P_J, the flag minor corresponding to its subset $J \in \{1,\dots,n\}$.

extended cluster: all chamber minors of a wiring diagram

~~?~~ cluster variables: the chamber minors for bounded chambers

frozen variables: the chamber minors for unbounded chambers

$2n-2$ of these

$\binom{n-1}{2}$ of these

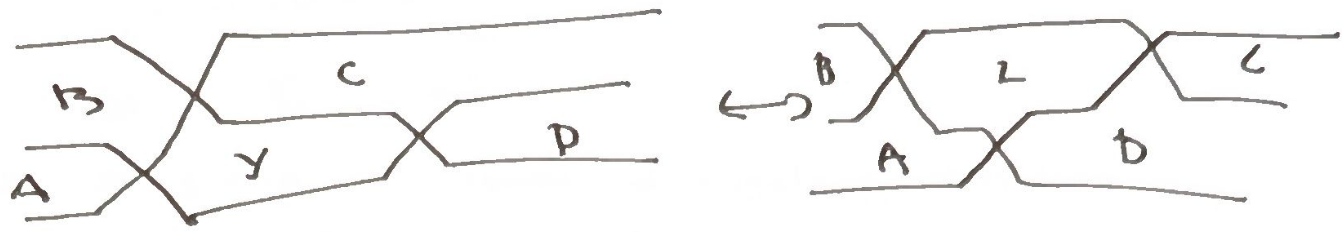
Thm Every flag minor can be written as a subtraction-free rat'l expr in the chamber minors of a given wiring diag.

Cor: If these $\frac{(n-1)(n+2)}{2}$ evaluate positively at a matrix $z \in SL_n$, then z is FTP.

pf: Follows by

(1) each flag minor appears as a chamber minor in some wiring diagram

(2) any two wiring diagrams can be transformed into each other by a sequence of local braid moves



(3) Under each braid move, collection of chamber minors changes by exchanging $Y \longleftrightarrow Z$, and have

$$YZ = AC + BD$$

Rmk: In fact, each flag minor can be written as a Laurent poly with pos. coeffs in the chamber minors of a given ~~wi~~ wiring diagram.

~~Lecture 3~~ ~~1/22/26~~

Put $G = SL_n$, $U < G$ subgroup of unipotent lower-triangulars

i.e. lower triangular matrices with 1's on diagonal

$U \curvearrowright G$ left multiplication action

$\rightsquigarrow U \curvearrowright \mathbb{C}[G]$ = ring of polynomials in the matrix entries of $A \in G$

$\mathbb{C}[G]^U$ = ring of U-invariant polynomials

i.e. $P \in \mathbb{C}[G]$ s.t. $P(yz) = P(z)$ $\forall$ $y \in U$, $z \in G$

Note: $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma c & \gamma d \end{pmatrix}$

i.e. $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} - v_1 - \\ - v_2 - \end{pmatrix} = \begin{pmatrix} - \alpha v_1 + \beta v_2 - \\ - \gamma v_2 - \end{pmatrix}$

Similarly, $\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & \varepsilon \\ 0 & 0 & \eta \end{pmatrix}\begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix} = \begin{pmatrix} - \alpha v_1 + \beta v_2 + \gamma v_3 - \\ - \delta v_2 + \varepsilon v_3 - \\ - \eta v_3 - \end{pmatrix}$, etc.

Def: The full flag variety in $\mathbb{C}^n$ is

$\left\{ \{0\} \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \;\middle|\; V_i \text{ is a subspace of dimension } i \text{ for } i = 1, \dots, n-1 \right\}$

This can be identified with the homogeneous space G/B, where $B < G$ is the subgrou

Lecture 3

1/23/26

Put $G = SL_n(\mathbb{C})$

$B \subset G$ subgroup of lower triangular matrices ← Borel subgroup

$U \subset G$ subgroup of $\underbrace{\text{unipotent}}_{\text{i.e. 1's on diagonal}}$ lower triangular matrices

Note:
$$\begin{pmatrix}\alpha & 0\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\alpha a & \alpha b\\ \beta a+\gamma c & \beta b+\gamma d\end{pmatrix},$$
i.e.
$$\begin{pmatrix}\alpha & 0\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}- v_1 -\\ - v_2 -\end{pmatrix} = \begin{pmatrix}- \alpha v_1 -\\ - \beta v_1+\gamma v_2 -\end{pmatrix}$$

Similarly,
$$\begin{pmatrix}\alpha & 0 & 0\\ \beta & \gamma & 0\\ \delta & \varepsilon & \zeta\end{pmatrix}\begin{pmatrix}- v_1 -\\ - v_2 -\\ - v_3 -\end{pmatrix} = \begin{pmatrix}- \alpha v_1 -\\ - \beta v_1+\gamma v_2 -\\ - \delta v_1+\varepsilon v_2+\zeta v_3 -\end{pmatrix} \quad \text{etc}$$

Def: The (full) flag variety in $\mathbb{C}^n$ is
$$\left\{ \{0\} \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset \mathbb{C}^n \;\middle|\; V_i \text{ is an } i\text{-dimensional subspace for } i=1,\dots,n-1 \right\}$$

Exercise. This is identified with the homogeneous space $B\backslash G$

Def: The basic affine space is $U\backslash G$

Note that we have $(\mathbb{C}^*)^n \hookrightarrow U\backslash G \twoheadrightarrow B\backslash G$, i.e.
the basic affine space is a torus bundle over the flag variety

Here $U \circlearrowright G$ action by left multiplication

$\rightsquigarrow U \circlearrowright \mathbb{C}[G]$ = ring of polynomials in the entries of $A \in SL_n$

$\mathbb{C}[G]^U$ = ring of U-invariant polynomials ← i.e. $P \in \mathbb{C}[G]$ s.t. $P(yz) = P(z)$ $\forall y \in U$, $z \in G$

Claim: ← by First and Second Fundamental Theorems of invariant theory

(1) the flag minors generate $\mathbb{C}[G]^U$

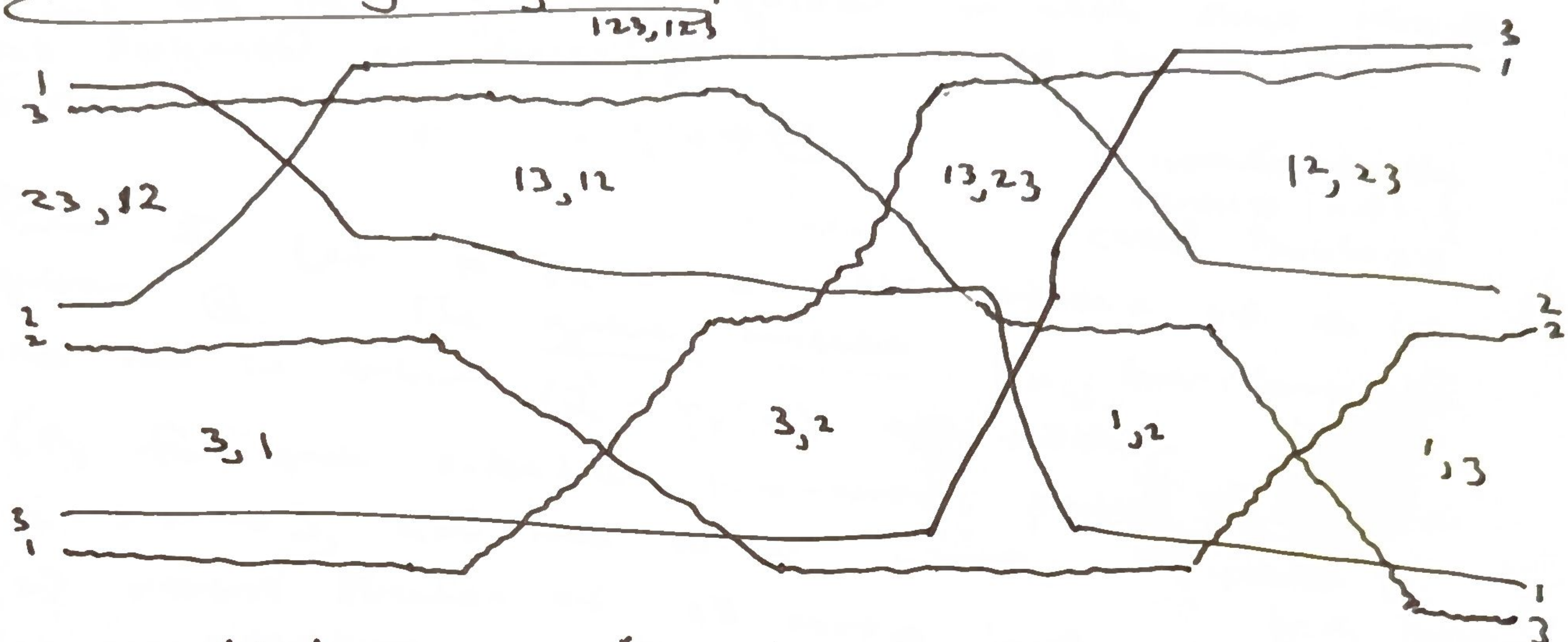
(2) the ideal of relations among flag minors is generated by quadratic relations called "generalized Plücker relations"

Checking TP for general $n \times n$ matrices

Given $I, J \subset \{1, \dots, n\}$ of same cardinality, put $\Delta_{I,J}$:= minor determined by rows in I and columns in J

Thus $Z \in Mat_{n \times n}$ is TP $\iff \Delta_{I,J}(Z) > 0$ for all $I, J \subset \{1, \dots, n\}$ with $|I| = |J|$

Double wiring diagrams:



$\rightsquigarrow$ chamber minors $\Delta_{3,1}, \Delta_{3,2}, \Delta_{1,2}, \Delta_{1,3}, \Delta_{23,12}, \Delta_{13,12}, \Delta_{13,23}, \Delta_{12,23}, \Delta_{123,123}$

Claim: number of chamber minors for a double wiring diagram is always n^2

Thm: Every minor of an $n \times n$ matrix can be written as a subtraction-free rational expression in the chamber minors of a given double wiring diagram.

Cor: Only need n^2 tests for positivity.

pf idea:

(1) every minor is a chamber minor for some double wiring diagram

(2) any two double wiring diagrams are related by sequence of local moves of three different kinds

(3) each local move results in an exchange of minors $Y \leftrightarrow Z$, where $YZ = AC + BD$.

Rmk: In fact in thm we really have Laurent polynomials with positive coefficients.

Rmk: The graph with vertices double wiring diagrams and edges local moves is not regular, but this will be rectified by the theory of cluster algebras.

Quivers and their mutations

Def: A *quiver* is a finite oriented graph with no loops or oriented 2-cycles.

Ex:

Def: An *ice quiver* is a quiver in which some vertices are designated as "frozen", and no arrows between two frozen vertices.

Ex: 

frozen

non-frozen ~~vert~~ vertices will be called "mutable"

Def: Let k be a mutable vertex of an ice quiver Q. The *quiver mutation* μ_k transforms Q into new ice quiver $Q' = \mu_k(Q)$ as follows:

(1) for each oriented two-arrow path $i \to k \to j$, add new arrow $i \to j$ (unless i, j both frozen)

(2) reverse direction of all arrows incident to k

(3) repeatedly remove any oriented 2-cycles until none left

Ex:

Exercise:

(1) mutation is an involution, i.e. $\mu_k(\mu_k(Q)) = Q$

(2) mutation commutes with reversing orientations of all arrows

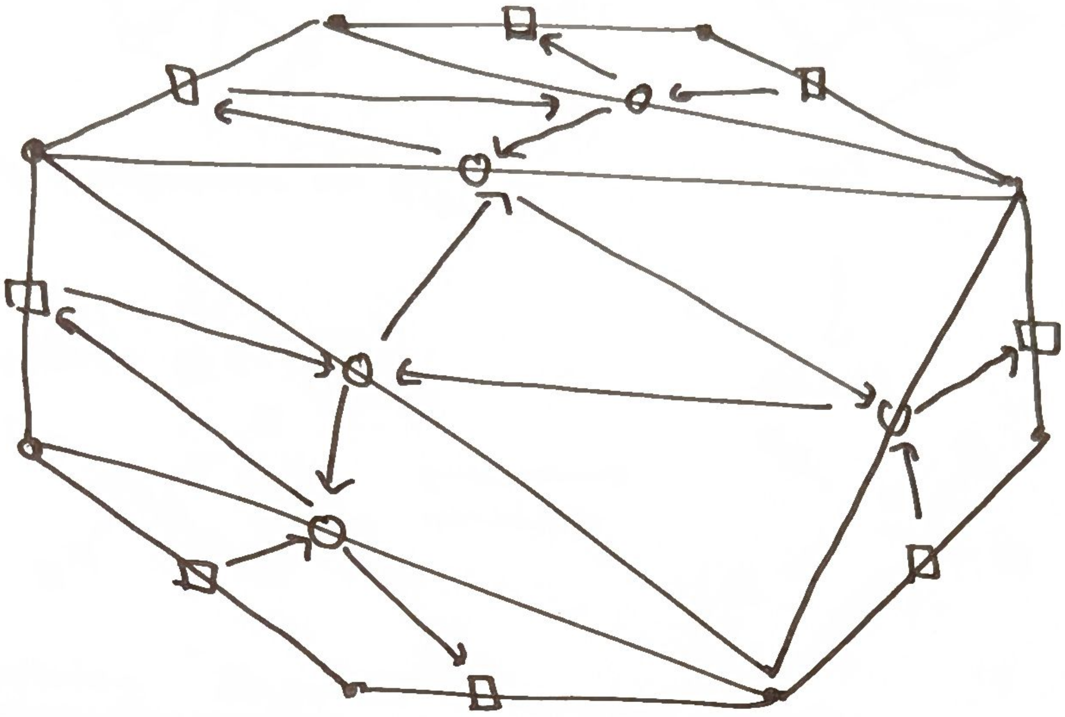
(3) if k, l mutable vertices with no arrows between them, then $\mu_l(\mu_k(Q)) = \mu_k(\mu_l(Q))$

Rmk: If k sink or source, μ_k simply reverses all arrows incident to k.

Exercise: If Q is a tree with no frozens, can get from any orientation to any other by a sequence of mutations at sinks and sources.

Triangulation ~~and~~ quiver

Can define a quiver from a ~~triangul~~ triangulation T of P_m.



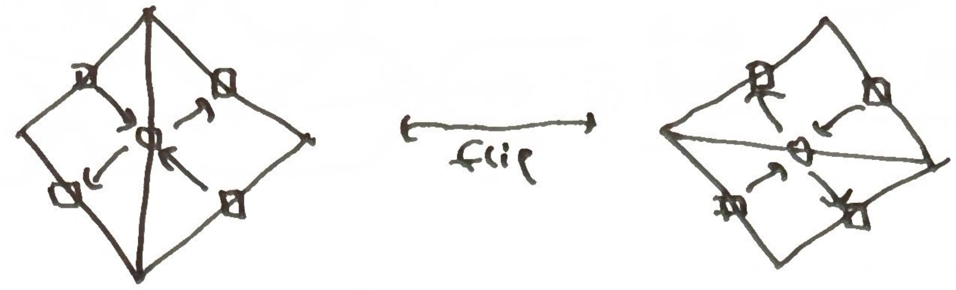
$Q(T)$

Exercise: If T is a triangulation of P_m and T' obtained by flip along diagonal γ, then $Q(T') = \mu_\gamma(Q(T))$

Lecture #4

1/26/26

Ex:

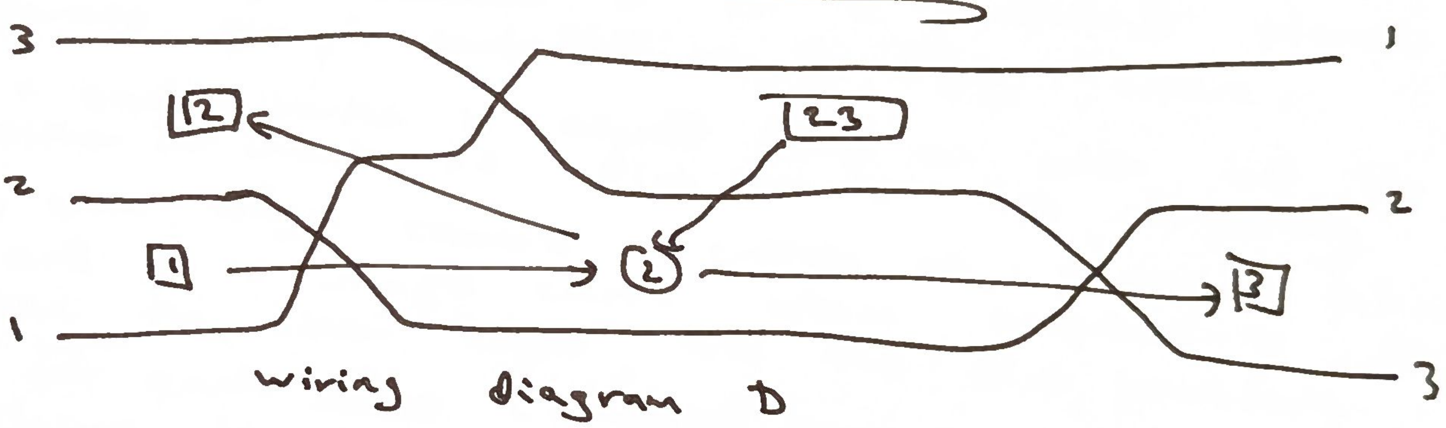


T = triangulation of $\mathbb{P}_4$ $\qquad$ T'

$\rightsquigarrow$ $\qquad$ $\rightsquigarrow$

quiver $Q(T)$ $\qquad$ $Q(T')$

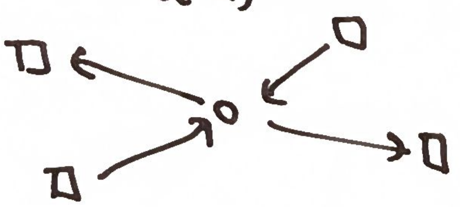
wiring diagram $\rightsquigarrow$ quiver



wiring diagram D

$\rightsquigarrow$

quiver $Q(D)$



Exercise: If D, D' wiring diagrams related by a braid move at chamber Y, then $Q(D') = \mu_Y(Q(D))$.

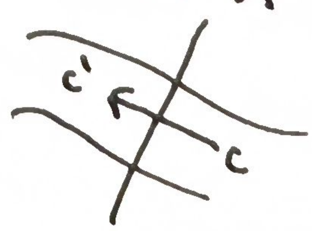
vertices: chambers of D (mutable if bounded, else frozen)

arrows: for chambers c, c', have $c \to c'$ in $Q(D)$ iff one of following holds.

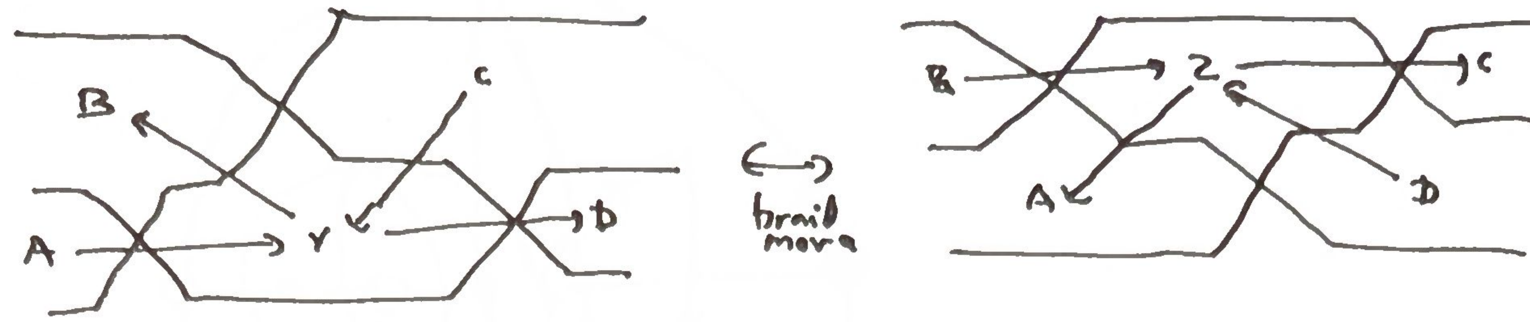
(i) right end of c = left end of c'

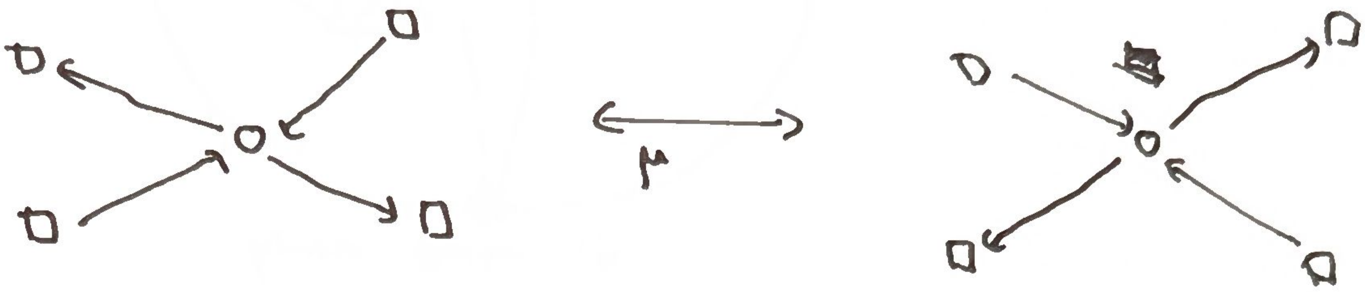
(ii) left end of c is directly above c', right end of c' is directly below c

(iii) left end of c is directly below c', right end of c' directly above c



Ex:

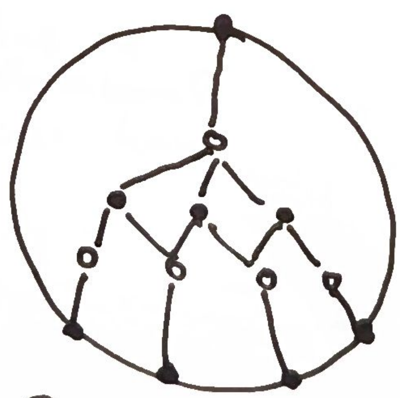




Rmk: Also have double wiring diagram D ⟶ quiver $Q(D)$. Description is more complicated, but special case of quiver associated to a planar bipartite graph.

Def: A plabic graph G is a connected planar bipartite graph embedded in a disk, where:

- each vertex is colored black or white and lies either in interior of disk or on its boundary
- each edge connects vertices of different colors and is a simple curve whose interior is disjoint from the other edges and the disk boundary
- for each face (connected cpnt of complement), the closure is simply connected
- each ~~internal~~ interior vertex has degree ≥ 2
- each boundary vertex has degree 1

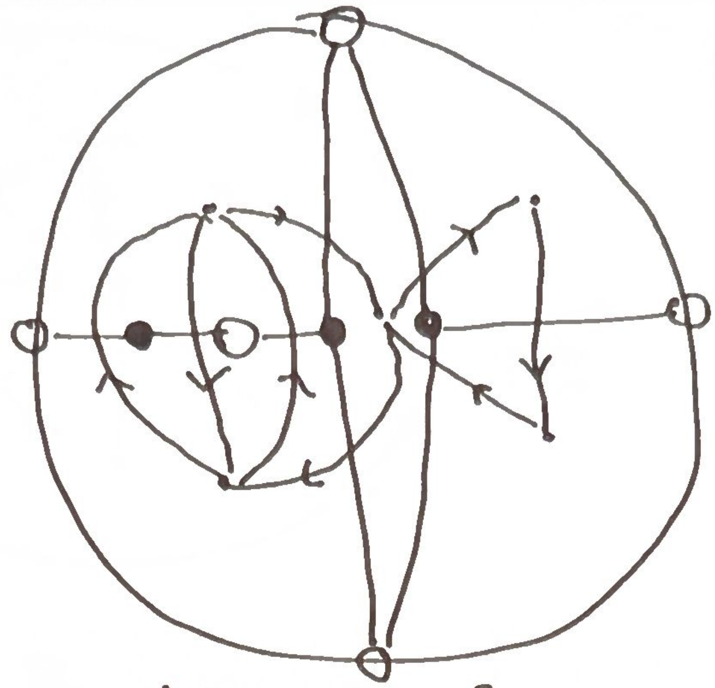


Note: we consider plabic graphs up to isotopy.

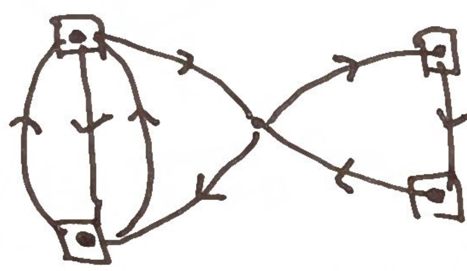
plabic graph G ⟶ quiver $Q(G)$

- vertices are faces of G (frozen if incident to disk boundary, else mutable)
- for each edge of G, have arrow joining the two faces it separates, using rule 
- remove oriented 2-cycles

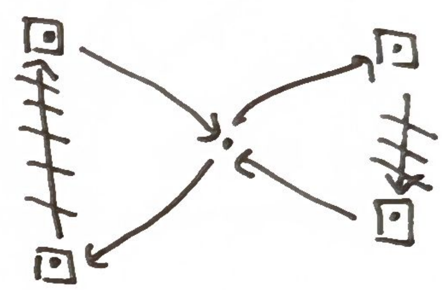
Ex:



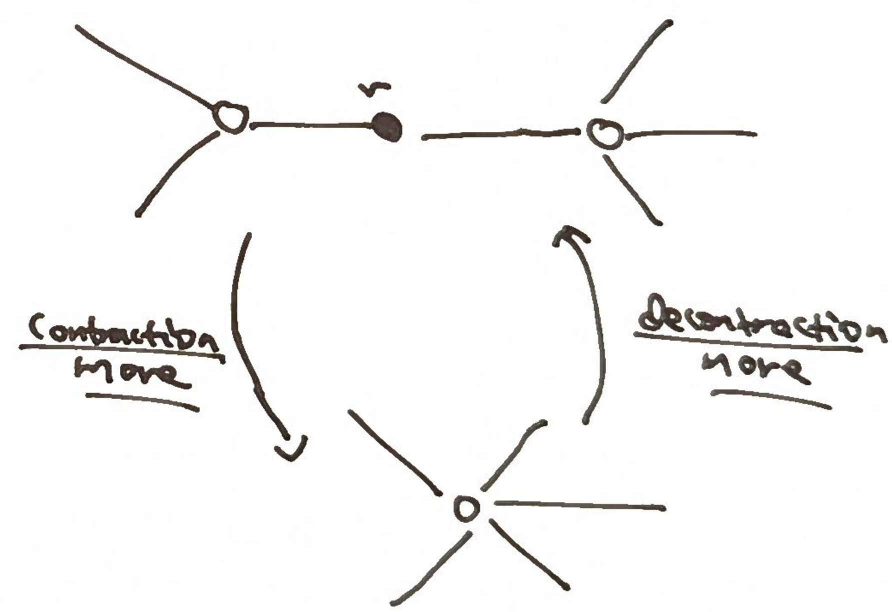
plabic graph G



remove oriented 2-cycles (and arrows between frozens)

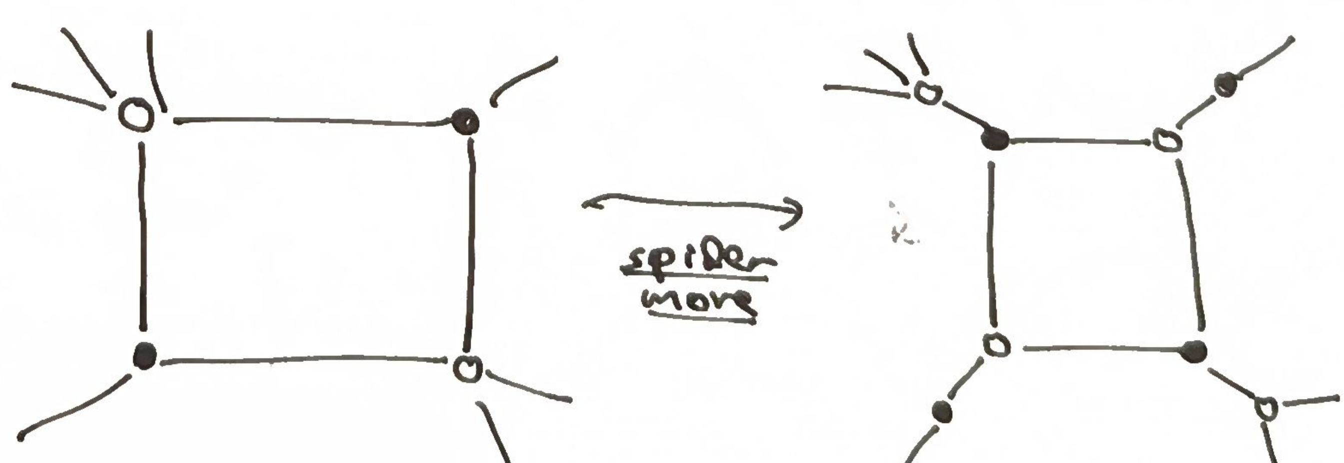


Def: Say v bivalent vertex adjacent to two interior vertices



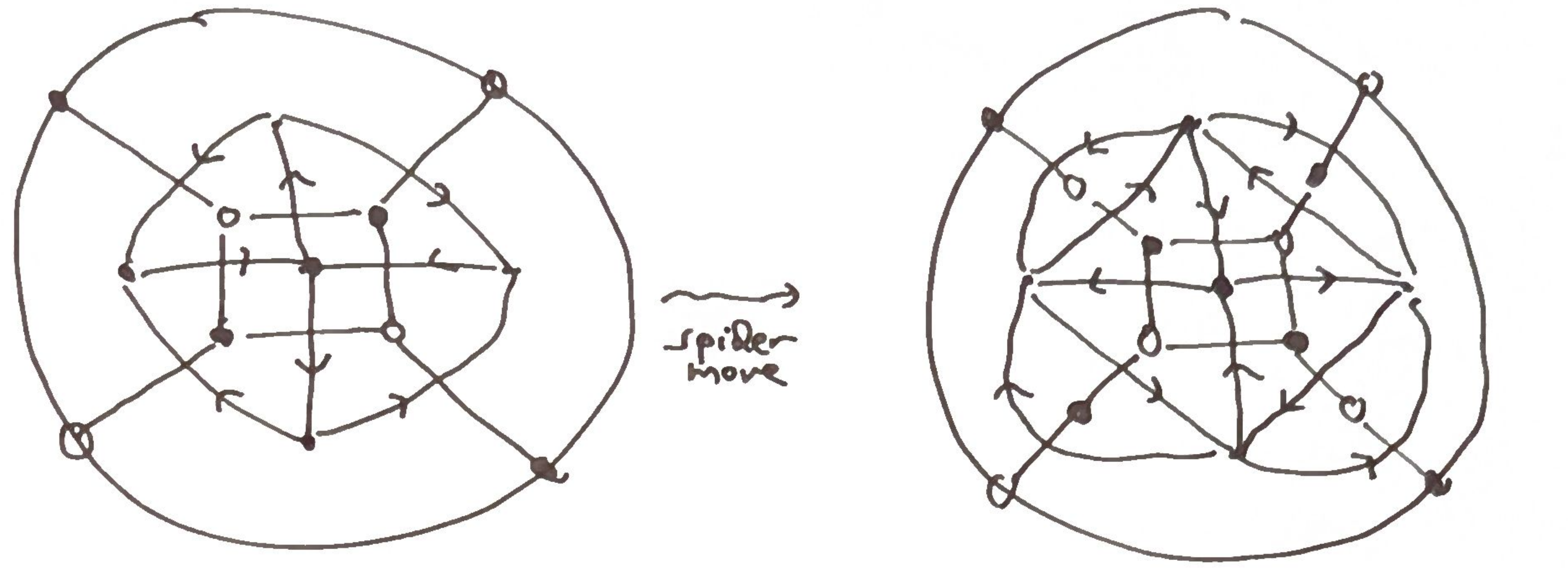
Rmk: does not change associated quiver

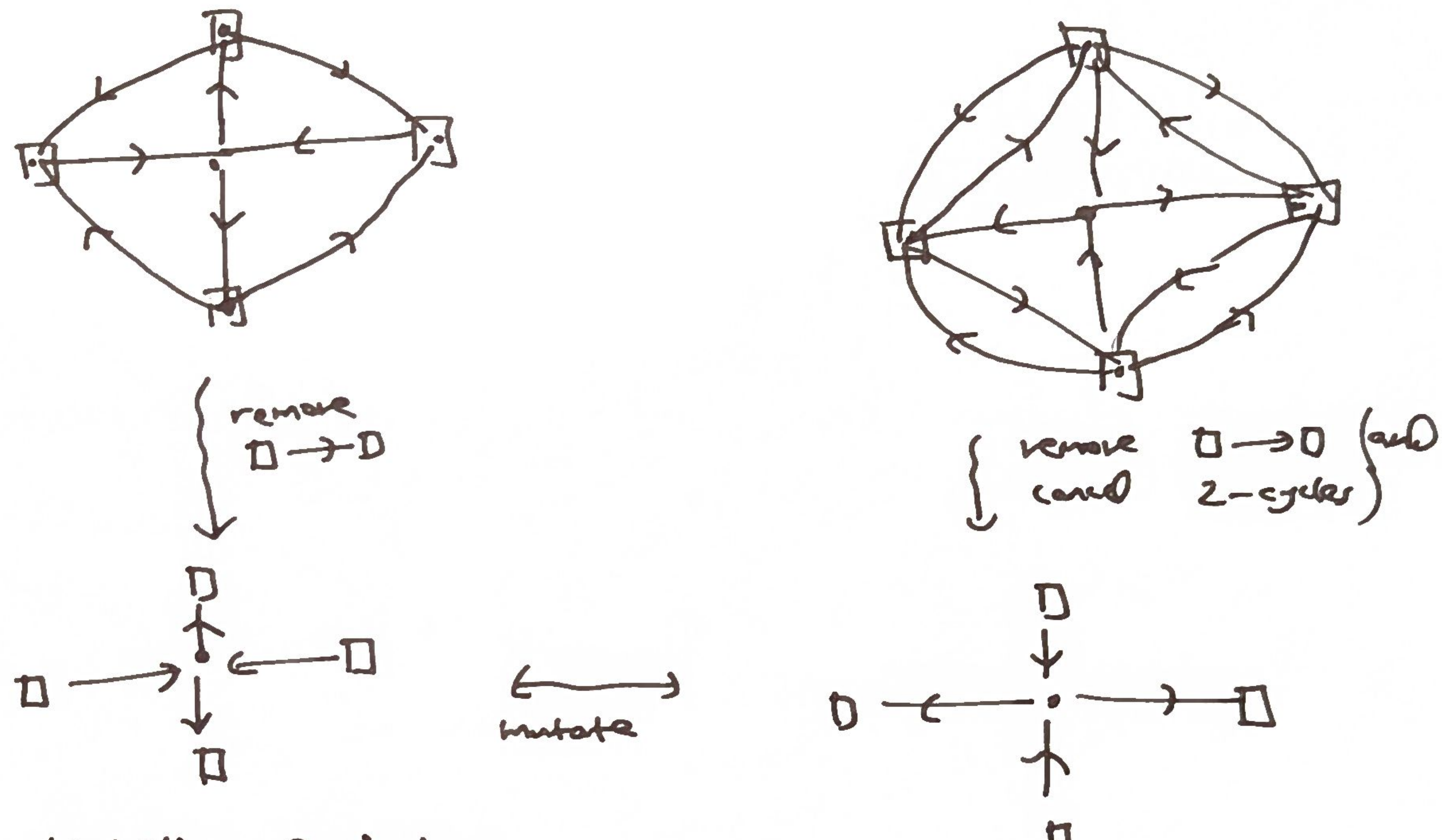
Def: Say G has a ~~quadrilat~~ quadrilateral face whose vertices have degree ≥ 3.



Exercise: If G, G' related by spider move, then ~~Q(G)~~ Q(G), Q(G') related by mutation

Ex



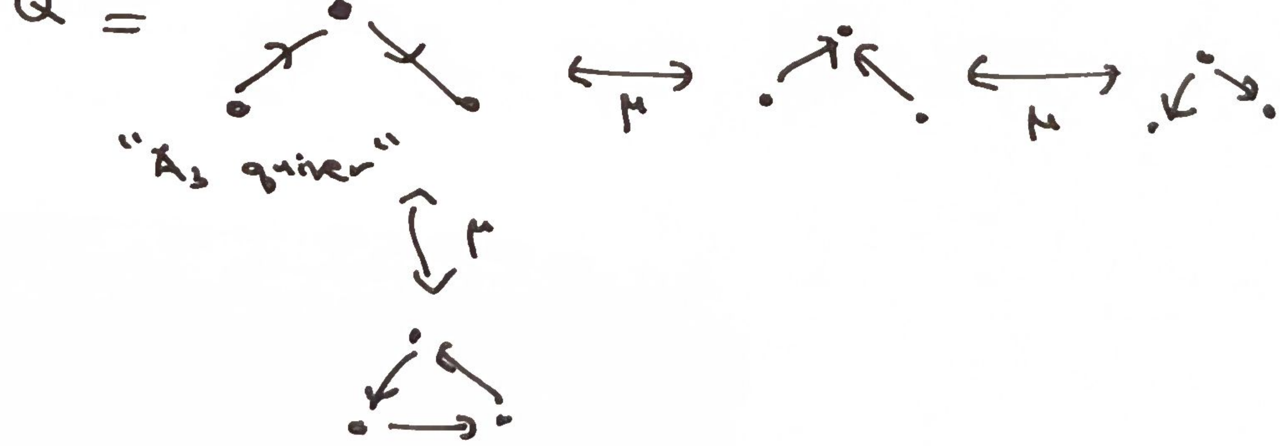


mutation equivalence

Def: Q, Q' mutation equivalent if Q becomes isomorphic to Q' after a sequence of mutations.

Put $[Q] :=$ set of ~~all~~ all quivers which are mutation equivalent to Q (up to isomorphism)

Ex: $Q =$



Exercise: $[Q]$ has 4 elements

Ex: $Q = $
$$\begin{tikzcd} & \bullet \arrow[dl, shift left] \arrow[dl, shift right] & \\ \bullet \arrow[rr, shift left] \arrow[rr, shift right] & & \bullet \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd} \quad \overset{\mu}{\longleftrightarrow} \quad \begin{tikzcd} & \bullet \arrow[dr, shift left] \arrow[dr, shift right] & \\ \bullet \arrow[ur, shift left] \arrow[ur, shift right] & & \bullet \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

"Markov quiver"

In fact, $[Q]$ is just a single element.

Def. Q has finite mutation type if $[Q]$ is finite.

Rmk: There is a classification theorem for quivers with no frozen vertices and finite mutation type.

Def: Q acyclic if no oriented cycles.

Thm (Caldero-Keller '06): If Q, Q' acyclic and mutation ~~eq~~ equivalent, then we can transform Q into Q' by a sequence of mutations at sources and sinks. In particular, Q, Q' have the same underlying undirected graphs.