Toric Geometry Seminar 1

Reference: [Fulton] An Introduction to Toric Varieties

Notation: A Lattice of dimension n $N \cong \mathbb{Z}^n$ and its dual Lattice $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$

Write $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$

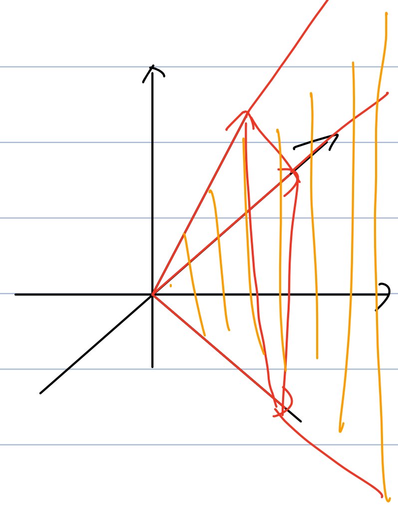
$N_\mathbb{R} \cong \mathbb{R}^n$, $M_\mathbb{R} \cong \mathbb{R}^n$

§ 1.2 Convex polyhedral cones (C.P.C.)

Definition: A convex polyhedral cone σ is the cone of a finite subset of $N_\mathbb{R}$

$$\sigma = \mathrm{cone}(S) := \{ r_1 v_1 + \cdots + r_s v_s \in N_\mathbb{R} \mid r_i \geq 0,\ v_i \in S \} \text{ for } S = \{v_1, \cdots, v_s\}$$

Examples:

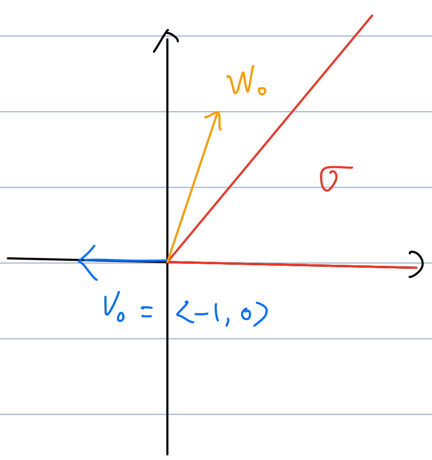


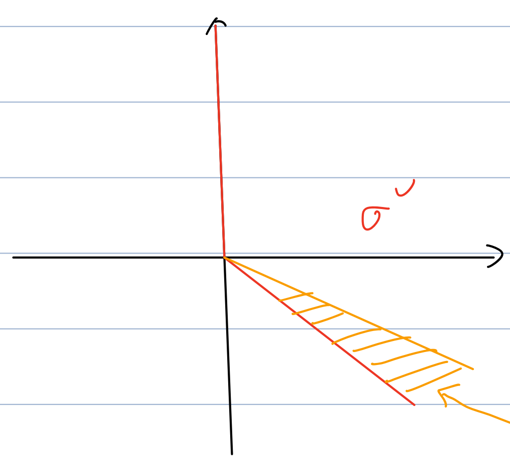
Def: The dimension of σ, $\dim(\sigma)$ is the dimension of the linear space spanned by σ.

Def: The dual of σ is defined by

$$\sigma^\vee = \{ u \in M_\mathbb{R} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}$$

Fact: (1) Given a C.P.C. σ and $v_0 \notin \sigma$, $\exists\, u_0 \in \sigma^\vee$ with $\langle u_0, v_0 \rangle < 0$.





Any $u \in \sigma^\vee$ works for v_0

choose u here for w_0

(2) $(\sigma^\vee)^\vee = \sigma$

(3) (Definition) A face τ of σ is the intersection of σ with any supporting hyperplane

$$\tau = \sigma \cap u^\perp = \{ v \in \sigma \mid \langle u, v\rangle = 0 \}$$

$\left(\text{of } u \in M_{\mathbb{R}}, \; u^\perp = \{ v \in N_{\mathbb{R}} \mid \langle u, v \rangle = 0 \} \right)$

A cone is a face of itself as $\sigma \cap 0^\perp = \sigma$; any other face is called a proper face.

(4) Any face is also a C.P.C.

(5) Any face of a face is a face.

$\tau = \sigma \cap u^\perp$, $\gamma = \tau \cap (u')^\perp$ for $u \in \sigma^\vee$ and $u' \in \tau^\vee$, then $\exists\, p > 0$ s.t.

$u' + pu \in \sigma^\vee$ and $\gamma = \sigma \cap (u' + pu)^\perp$

$\langle u' + pu, v\rangle = \langle u', v\rangle + p\langle u, v\rangle$

LHS $\subset$ RHS if $v \in \gamma = \tau \cap (u')^\perp$, $\langle u, v\rangle = 0$, $\langle u', v\rangle = 0$

RHS $\subset$ LHS if $v \in \sigma \cap (u' + pu)^\perp$, $\langle u' + pu, v\rangle = \langle u', v\rangle + p\langle u, v\rangle = 0$

(Choose p large enough w/ $\langle u', w\rangle + \langle pu, w\rangle \geq 0$ for all generating elements w.

This forces $v \in \tau$ b/c if $\langle u', v\rangle > 0$ then $\langle u' + pu, v\rangle > 0$

$\Rightarrow \langle u, v\rangle = 0 \Rightarrow \langle u', v\rangle = 0$.

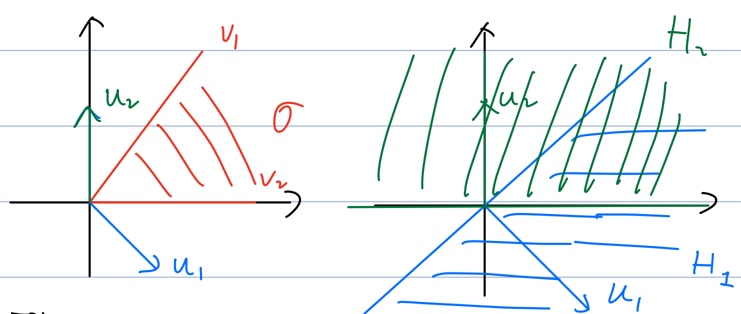
(5) Any face is contained in a codimension-1 face i.e. a facet.

(6) Any face is the intersection of all facets containing it.

(7) The topological boundary of a cone that spans $N_{\mathbb{R}}$ is the union of its proper faces

(8) If σ spans $N_{\mathbb{R}}$ and $\sigma \neq N_{\mathbb{R}}$, then σ is the intersection of the half-spaces

$H_\tau = \{ v \in V : \langle u_\tau, v\rangle \geq 0 \}$ as τ ranges over the facets of σ. where u_τ is the normal direction to τ as a hyperplane.



$H_1 \cap H_2 = \sigma$

(9) The dual of a C.P.C. is a C.P.C.

(Gordan's Lemma) If σ is a rational C.P.C., (Def: $\sigma = \text{Cone}(S)$ with $S \subset N \subset N_{\mathbb{R}}$), then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.

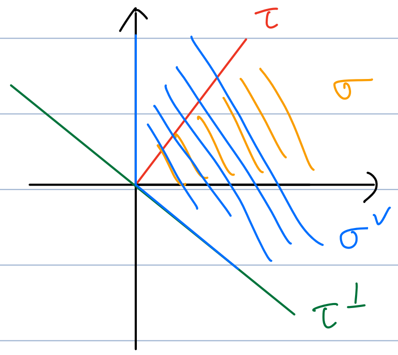
Pf: Let $u_1, \dots, u_s$ generate $\sigma^\vee$ as a cone. Let $K = \{\sum t_i u_i \mid 0 \le t_i \le 1\}$

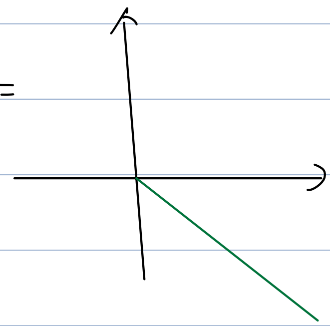
Since K is compact and M is discrete, $K \cap M$ is finite. Then $K \cap M$ generates S_σ as follows: If $u \in \sigma^\vee \cap M$, $u = \sum r_i u_i$ for $r_i \ge 0$. $\exists m_i \in \mathbb{Z}_{\ge 0}$ s.t. $r_i = m_i + t_i$ w/ $t_i \in [0,1]$ Now $u = \underbrace{\sum m_i u_i}_{\in \langle K \cap M \rangle} + \underbrace{\sum t_i u_i}_{\in K}$

(10) If τ is a face of σ, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$

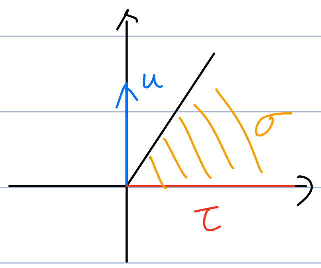
$\dim(\tau) + \dim(\sigma^\vee \cap \tau^\perp) = n = \dim(V)$.

This sets up a one-to-one correspondence btw the faces of σ and the faces of $\sigma^\vee$



$\sigma^\vee \cap \tau^\perp =$  is a face of $\sigma^\vee$

(11) If $u \in \sigma^\vee$, and $\tau = \sigma \cap u^\perp$, then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\ge 0} \cdot (-u)$



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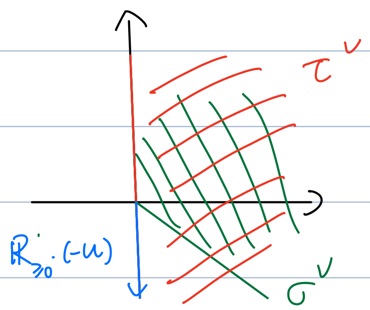


(Proposition 2) Let σ be a rational convex polyhedral cone, and let u be in $S_\sigma = \sigma^\vee \cap M$. Then $\tau = \sigma \cap u^\perp$ is a rational convex polyhedral cone. All faces of σ have this form and $S_\tau = S_\sigma + \mathbb{Z}_{\ge 0} \cdot (-u)$

Pf: If τ is a face, then $\tau = \sigma \cap u^\perp$ for any u in the relative interior of $\sigma^\vee \cap \tau^\perp$ and u can be taken from M since $\sigma^\vee \cap \tau^\perp$ is rational. Given $w \in S_\tau$ $w + pu$ is in $\sigma^\vee$ for $p \gg 0$ $\Rightarrow$ $w \in S_\sigma + \mathbb{Z}_{\ge 0} \cdot (-u)$

(12) If σ & σ' are c.p.c. whose intersection τ is a face of each, then there is a u in $\sigma^\vee \cap (-\sigma')^\vee$ with $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$

(Proposition 3) If σ, σ' r.c.P.C. whose intersection is a face of each, then $S_\tau = S_\sigma + S_{\sigma'}$

(13) For a C.P.C. σ, the following conditions are equivalent

(i) $\sigma \cap (-\sigma) = (0)$;

(ii) σ contains no nonzero linear subspaces;

(iii) there is a u in $\sigma^\vee$ w/ $\sigma \cap u^\perp = (0)$;

(iv) $\sigma^\vee$ spans M

Definition: A strongly convex polyhedral cone is a C.P.C. satisfying (13).

Theorem: σ n-dim'l C.P.C. s.t. $\sigma \neq \mathbb{R}^n$. Let facets of σ be $\tau_i = \sigma \cap u_i^\perp$

then $\sigma = \bigcap_{i=1}^{s} H_{u_i}$ for all inward normal u_i for each facet τ_i

and $\sigma^\vee = \mathrm{cone}(u_1, \cdots, u_s)$

1.3 Affine Toric Varieties.

Let σ be a strongly convex rational polyhedral cone, $S_\sigma = \sigma^\vee \cap M$ is a f.g. ~~semigroup~~ monoid

One can form the corresponding "group ring" $\mathbb{C}[S_\sigma]$, which is a commutative algebra.

If V is the generating set of S_σ, then we can write

$$\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^u]_{u\in V} \quad \text{as a polynomial ring where}$$
$$\chi^u \cdot \chi^{u'} = \chi^{u+u'} \quad \text{with unit } 1 = \chi^0$$

Recall that a finitely generated commutative $\mathbb{C}$ algebra A determines a complex affine variety $X = \mathrm{Spec}(A)$ w/ underlying space the spectrum of A

Def: For σ as above, $V_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is the affine toric variety associated to σ.

Let $N = M = \mathbb{Z}^n$ w/ standard basis $e_1, \cdots, e_n$

Examples:

(1) $\sigma = \mathrm{cone}(e_1, \cdots, e_n) \subset \mathbb{R}^n$ is self dual. $\sigma^\vee = \mathbb{Z}^n_{\geq 0}$ the resulting algebra is then $\mathbb{C}[x_1, \cdots, x_n] \Rightarrow V_\sigma = \mathbb{C}^n$.

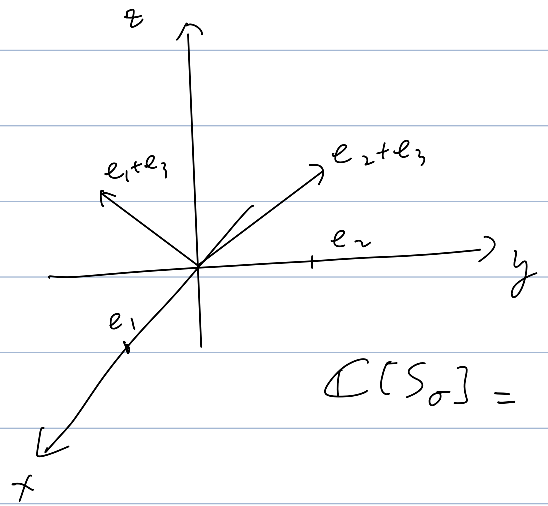
(2) $\sigma = \mathrm{cone}(e_1, \cdots, e_d) \subset \mathbb{R}^n$ for $d<n$, $\sigma^\vee = \mathrm{cone}(e_1, \cdots, e_d, \pm e_{d+1}, \cdots, \pm e_n)$

$\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \cdots, x_d, x_{d+1}^{\pm 1}, \cdots, x_n^{\pm 1}] \Rightarrow V_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$

(3) $\sigma = \{0\} \Rightarrow \sigma^\vee = \mathbb{R}^n \Rightarrow \mathbb{C}[S_\sigma] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \Rightarrow V_\sigma = (\mathbb{C}^*)^n$

(4) $\sigma = \mathrm{cone}(e_1, e_2, e_1+e_3, e_2+e_3)$

The inward normal for each facet is given by



$$e_2 \times (e_2+e_3) = u_1 = (1,0,0)$$
$$(e_1+e_3)\times e_1 = u_2 = (0,1,0)$$
$$e_1 \times e_2 = u_3 = (0,0,1)$$
$$(e_2+e_3)\times(e_1+e_3) = u_4 = (1,1,-1)$$

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x, y, z, xyz^{-1}] \cong \mathbb{C}[u,v,z,w]/(uv = zw)$$

Definition: A toric variety is an normal variety such that

(1) $(\mathbb{C}^*)^n$ is a Zariski open subset of X

(2) the action of $(\mathbb{C}^*)^n$ extends to an action of $(\mathbb{C}^*)^n$ on X.

Theorem 1: An affine variety X that is toric iff $\exists$ s.c.R.P.C. σ s.t. $X = V_\sigma$

Corollary: For a s.c.R.P.C. σ, the affine toric variety V_σ is indeed a toric variety.

Fulton definition

Nonexample: $X = \mathrm{Spec}(\mathbb{C}[x,y]/(y^2-x^3))$ is not an affine toric variety.

One may identify $\mathbb{C}[x,y]/(y^2-x^3) \cong \mathbb{C}[t^2,t^3] = \mathbb{C}[S]$

for the semigroup $S = \{0,2,3,4,5,6,\dots\} \subset \mathbb{Z}$.

but X is not normal because in $\mathrm{Frac}(\mathbb{C}[x,y]/(y^2-x^3))$,

the equation $t^2 - x = 0$ has a solution $t = y/x$ but $y/x \notin \mathbb{C}[x,y]/(y^2-x^3)$

Thus X is not normal.

Recall last time:

For any S.C.R.P.C. σ, $\exists$ an affine toric variety $V_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$

We know that if $\tau \subset \sigma$ is a face, then $S_\tau \supset S_\sigma \Rightarrow \exists\ \underset{\overset{\|}{A_\sigma}}{\mathbb{C}[S_\sigma]} \xrightarrow{\phi} \underset{\overset{\|}{A_\tau}}{\mathbb{C}[S_\tau]}$

Lemma: ϕ induces an open embedding $f: V_\tau \to V_\sigma$ as a principal open subset.

Pf: proposition from last time said $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$ for $\tau = \sigma \cap u^\perp$

$\Rightarrow$ All elements in $\mathbb{C}[S_\tau]$ is of the form $\chi^{w-pu} = \chi^w / \chi^{pu}$ for $w \in S_\sigma$ and $p \geq 0$.

$\Rightarrow A_\tau = (A_\sigma)_{\chi^u}$

Observation: σ S.C.R.P.C. $\Rightarrow V_\sigma$ contains $(\mathbb{C}^*)^n$ as a dense open subset

① Gordan's lemma $\Rightarrow$ $m_1, \cdots, m_r$ generates $\sigma^\vee \cap M$.

$\Rightarrow \mathbb{C}[x_1, \cdots, x_r] \xrightarrow{\alpha} \mathbb{C}[S_\sigma] \xrightarrow{\beta} \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] =: \mathbb{C}[M]$

α is quotienting by all relations among generators

β is inclusion into the field of fraction.

Note we can always pick $m_0 \in \mathrm{int}(\sigma^\vee)$ as σ is strongly convex

$\forall\ m \in M$, $\exists\ l$ large s.t. $m + l m_0 \in \sigma^\vee$ as in the pf from last time.

(i.e. make $l m_0$ evaluation at generators of σ large so that $m + l m_0 \geq 0$ on them)

$\Rightarrow \underset{\mathbb{C}[M]}{t^m} = \dfrac{t^{m+lm_0}}{(t^{m_0})^l} \in \mathbb{C}[S_\sigma]_{t^{m_0}}$

($\forall\ m = \sum_{i=1}^n r_i e_i$, $t^m = t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$ a rational function)

$\Rightarrow \beta$ factors through $\mathbb{C}[S_\sigma] \longrightarrow \mathbb{C}[S_\sigma]_{t^{m_0}} = \mathbb{C}[M] \simeq \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$

m_0 has nonzero component on each standard basis vector e_i thus can give all $t_i^{\pm 1}$ by clearing the other factors by multiplication

principal open dense subset $\cong (\mathbb{C}^*)^n$ in V_σ

② $\underset{T}{(\mathbb{C}^*)^n} \longrightarrow \mathbb{C}^r$

$t = (t_1, \cdots, t_n) \longmapsto (t^{m_1}, \cdots, t^{m_r})$

The action of T on $\mathbb{C}^r$ is induced by the map above and this extends to an action of T on V_σ natural as the kernel of α is stable under T action.

$\forall\ f \in \ker(\alpha)$ $\quad f = \prod_{i=1}^r x_i^{a_i} - \prod_{j=1}^r x_j^{b_j}$ where the tuple

$a_i \quad b_j$ satisfies $\quad a_1 m_1 + \cdots + a_r m_r = b_1 m_1 + \cdots + b_r m_r$

$$\Rightarrow \quad t \cdot f = \prod_{i=1}^{r} (t^{m_i} x_i)^{a_i} - \prod_{j=1}^{r} (t^{m_j} x_j^{b_j}) = 0$$

On algebra level this action corresponds to $\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$

$$x^u \longmapsto x^u \otimes \chi^u$$

Further example:

(Product) if $\sigma \subset N$, $\sigma' \subset N'$ s.c.R.P.C.'s, then $\sigma \times \sigma' \subset N \oplus N'$ s.c.R.P.C.

algebra: $\mathbb{C}[S_{\sigma \times \sigma'}] \simeq \mathbb{C}[S_\sigma] \otimes \mathbb{C}[S_{\sigma'}]$

geometry: $V_{\sigma \times \sigma'} \cong V_\sigma \times V_{\sigma'}$

§1.4 Fans and Toric Varieties

<u>Def</u>: A Fan $\Sigma \in N$ is a collection of cones σ in $N_{\mathbb{R}}$ such that

(1) Each face of a cone in Σ is also a cone in Σ.

(2) The intersection of two cones in Σ is a face of each.

Next, we want to construct a variety out of a fan via gluing V_σ for $\sigma \in \Sigma$ over V_τ where τ runs over all faces given by intersections of σ

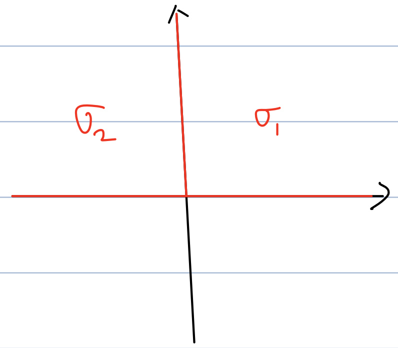
Examples:

$\Sigma_0 = \{ \underset{\sigma_1}{\mathrm{cone}(e_1)}, \underset{\sigma_2}{\mathrm{cone}(-e_1)} \}$ $\qquad \tau = \sigma_1 \cap \sigma_2 = \{0\}$

$V_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[x]) \qquad V_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[x^{-1}]) \qquad V_\tau = \mathrm{Spec}(\mathbb{C}[x, x^{-1}])$

$$\begin{array}{ccc} & & \mathbb{C}[x^{-1}] \\ & & \downarrow \\ \mathbb{C}[x] & \longrightarrow & \mathbb{C}[x, x^{-1}] \end{array} \qquad \begin{array}{ccccc} \mathbb{P}^1 & \longrightarrow & \mathbb{C} & & x^{-1} \\ \downarrow & & \uparrow & & \uparrow \\ \mathbb{C} & \longleftarrow & \mathbb{C}^* & & x \\ x & \longleftarrow\!\shortmid & x & & \end{array}$$

Σ_1



$\Sigma_1 = \{\sigma_1, \sigma_2\}$ $\quad \mathrm{Spec}(\mathbb{C}[a,b]) =: V_{\sigma_1} \cong \mathbb{C}^2 \cong V_{\sigma_2} := \mathrm{Spec}(\mathbb{C}[s,t])$

$\sigma_1 \cap \sigma_2 = \tau = \mathrm{cone}(e_2)$ $\quad V_\tau = \mathrm{Spec}(\mathbb{C}[x^{\pm 1}, y])$

On algebra level we have $\quad \mathbb{C}[a,b] \hookrightarrow \mathbb{C}[x^{\pm 1}, y] \hookleftarrow \mathbb{C}[\overset{x^{-1}}{\overset{\shortparallel}{s}}, \overset{y}{\overset{\shortparallel}{t}}]$

$\therefore$ geometry $\therefore$ $\quad \mathbb{C}_a \times \mathbb{C}_b \hookleftarrow \mathbb{C}^*_x \times \mathbb{C}_y \hookrightarrow \mathbb{C}_s \times \mathbb{C}_t$

$\mathbb{C}_a$ and $\mathbb{C}_s$ is glued over $\mathbb{C}^*$ via the transition maps

$$\mathbb{C}_a \hookleftarrow \mathbb{C}^*_x \hookrightarrow \mathbb{C}_s$$

$$x \longleftarrow\!\shortmid\; x \;\longmapsto x^{-1}$$

i.e. over $\mathbb{C}^*_x$ $\quad a = \frac{1}{s}$.

This is exactly the standard 2 covering pieces of $\mathbb{P}^1$.

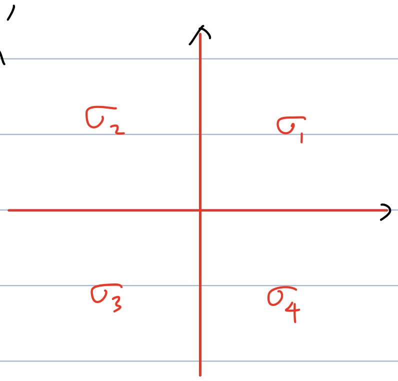
i.e. $V_{\Sigma_1} \cong \mathbb{P}^1 \times \mathbb{C}$

Note here $\Sigma_1 = \Sigma_0 \times \{\tau\}$ for $\tau = \mathrm{cone}(e_2)$

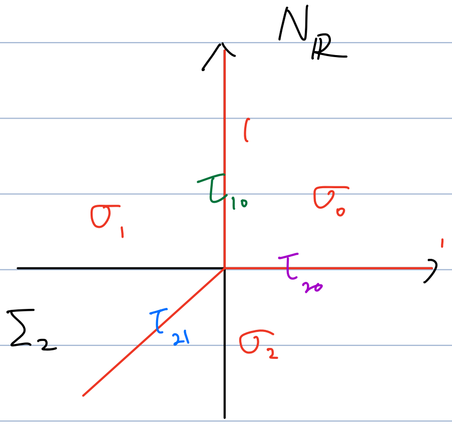
$$V_{\Sigma_1} = V_{\Sigma_0} \times V_{\{\tau\}}$$

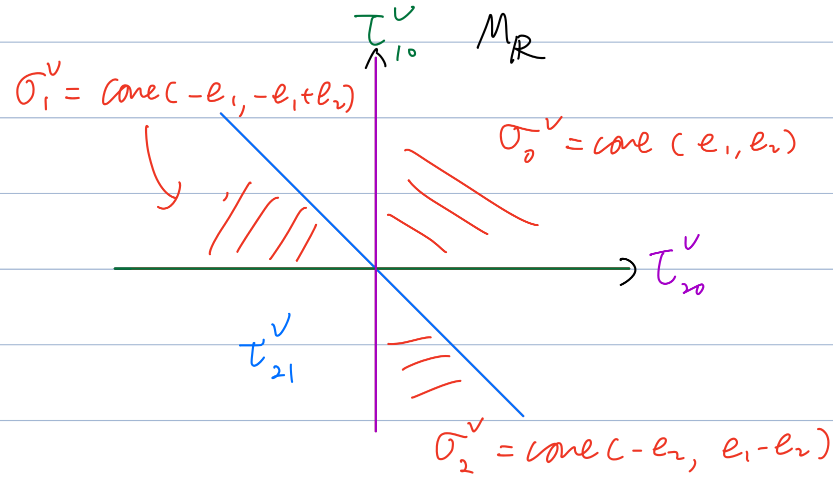
as a toric variety associated to $\tau \subset \mathbb{R}$

Σ_1'



$$V_{\Sigma_1'} \text{ "}=\text{" } \mathbb{P}^1 \times \mathbb{C}_y \underset{\substack{\mathbb{C}^* \\ y \mapsto y^{-1}}}{\cup} \mathbb{P}^1 \times \mathbb{C}_{y^{-1}} = \mathbb{P}^1 \times \mathbb{P}^1$$





$V_{\sigma_0} := \mathrm{Spec}(\mathbb{C}[x,y])$

$V_{\sigma_1} := \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y])$

$V_{\sigma_2} := \mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}])$

Let A_σ be the ring $\mathbb{C}[S_\sigma]$ for a cone σ.

$$\begin{tikzcd}
& A_{\sigma_1} = \mathbb{C}[x^{-1}, x^{-1}y] \arrow[dl] \arrow[dr] & \\
\mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}} \cong A_{\tau_{20}} \cong \mathbb{C}[x,y]_x & & \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \cong A_{\tau_{21}} \cong \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}} \\
A_{\sigma_0} = \mathbb{C}[x,y] \arrow[u] \arrow[dr] & & A_{\sigma_2} = \mathbb{C}[xy^{-1}, y^{-1}] \arrow[u] \arrow[dl] \\
& \mathbb{C}[x,y]_y \cong A_{\tau_{20}} \cong \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}} &
\end{tikzcd} \qquad (\star)$$

Now consider the homogeneous coordinates $[x_0 ; x_1 ; x_2]$ on $\mathbb{P}^2$

Then $\begin{array}{l} x \mapsto \frac{x_1}{x_0} \\ y \mapsto \frac{x_2}{x_0} \end{array}$ identifies "Spec" of $(\star)$ with the standard covering of $\mathbb{P}^2$

by identifying V_{σ_0} with $\{x_0 \neq 0\}$, V_{σ_1} with $\{x_1 \neq 0\}$, V_{σ_2} with $\{x_2 \neq 0\}$

Extra example Let $v_1 = (0,1)$ $v_2 = (1,0)$ $v_3 = (0,-1)$ $v_4 = (-1,n)$

$\sigma_1 = \mathrm{cone}(v_1, v_2)$ $\sigma_2 = \mathrm{cone}(v_2, v_3)$ $\sigma_3 = \mathrm{cone}(v_3, v_4)$ $\sigma_4 = \mathrm{cone}(v_4, v_1)$

$\Sigma_{(n)} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$

Note $\Sigma_{(1)} = \Sigma_1' \Rightarrow X_{\Sigma_{(1)}} = \mathbb{P}^1 \times \mathbb{P}^1$

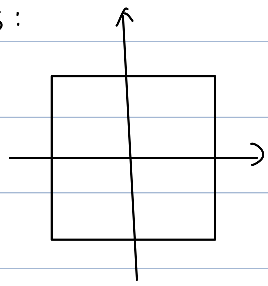
In general $X_{\Sigma_{(n)}}$ is denoted by F_n called Hizebruch surface

§1.5 Toric Varieties From Polytopes

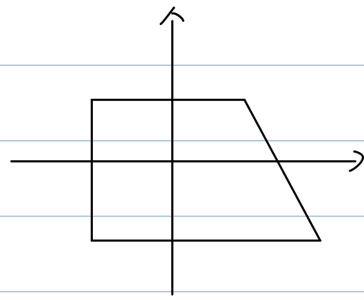
Definition: A convex polytope P in a finite dimensional vector space E is the convex hull of finite subset S of E. For $u \in E^*$, the supporting hyperplane $H_{u,r} = \{v \in E \mid \langle u,v \rangle = r\}$ for some $r \in \mathbb{R}$ intersects with P to give a face of P for $P \subset H^+_{u,r} = \{v \in E \mid \langle u,v \rangle \geq r\}$. Again, codim 1 faces are called facets and 0-dim'l faces are vertices. For our purpose, we restrict to the case of $E = M_{\mathbb{R}} = N_{\mathbb{R}}^\vee \simeq \mathbb{R}^n$ and $S \subset M$ finitely many lattice point. Such a P is called a lattice polytope.

Rmk: we assume K is of dimension n (*)

Examples:



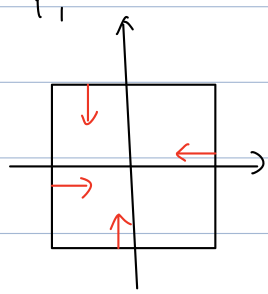
$P_1 = \mathrm{conv}((1,1), (1,-1), (-1,1), (-1,-1))$



$P_2 = \mathrm{conv}((1,1), (2,-1), (-1,1), (-1,-1))$

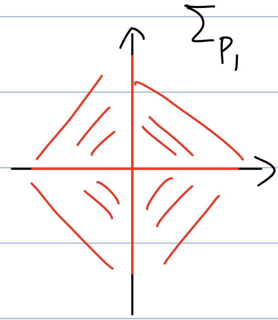
Definition / Construction (Normal Fan): Given a polytope P. There's an associated normal fan $\Sigma_P = \{\sigma_F \mid F \text{ a face of } P\}$ where σ_F is the cone generated by all inward normals to each facet containing F. To Σ_P, we have a toric variety $X_P := X_{\Sigma_P}$.

P_1

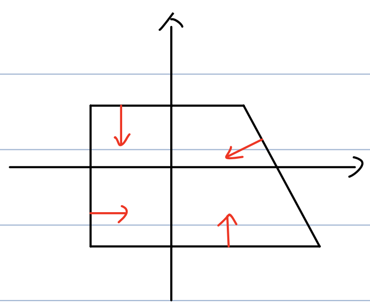


$\rightsquigarrow$

Σ_{P_1}



$\rightsquigarrow$ $X_{P_1} = \mathbb{P}^1 \times \mathbb{P}^1$



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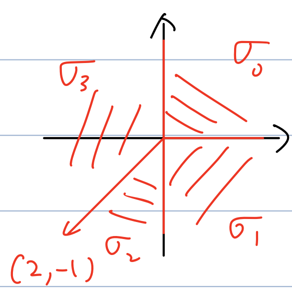
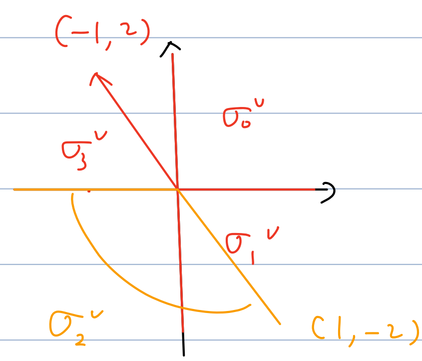


$\rightsquigarrow$



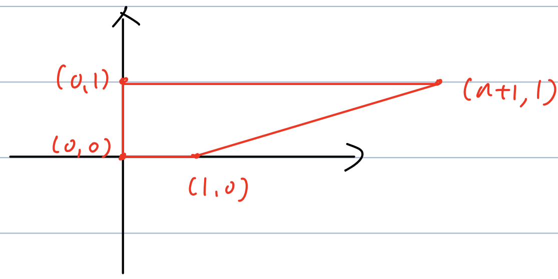
$X_{\sigma_0} := \mathrm{Spec}(\mathbb{C}[x,y]) \cong \mathbb{C}^2 \cong \mathrm{Spec}(\mathbb{C}[x,y^{-1}]) =: X_{\sigma_1}$

$X_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[x^{-1}, y^{-1}, xy^{-2}]) \cong \mathrm{Spec}(\mathbb{C}[u,v,w]/(uw-v^2))$

$X_{\sigma_3} = \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y, x^{-1}y^2]) \cong \mathrm{Spec}(\mathbb{C}[a,b,c]/(ac-b^2))$

are singular but not "too bad"

Going back to $\Sigma_{(n)} \rightsquigarrow P_n$



Theorem : The toric variety of a fan Σ in $N_{\mathbb{R}}$ is projective

$\iff$ Σ is the normal fan of an n-dimensional lattice polytope P in $M_{\mathbb{R}}$.

Recall a projective variety is the vanishing locus of some homogeneous polynomial on projective space

Toric Geometry 3

§ 2.1 Local properties of toric varieties

e.g. $\sigma = \mathbb{R}^n_{\geq 0} \Rightarrow U_\sigma = \mathbb{C}^n = \operatorname{Spec}(\mathbb{C}[x_1, \cdots, x_n]) \hookleftarrow T^n = (\mathbb{C}^*)^n$ $\qquad \mathbb{R}^k_{\geq 0}$

There're other strata of the form $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ associated to faces of σ.

In particular, T^n is the smallest stratum where all monomials x_i's are invertible.

The remaining part $U_\sigma \setminus T^n = x_\sigma = (0, 0, \cdots, 0)$ is called the distinguished point.

Given $\sigma \subset N_\mathbb{R} \Rightarrow S_\sigma = \sigma^\vee \cap M$ and $U_\sigma = \operatorname{Spec}(\mathbb{C}[S_\sigma])$

The coordinate ring $\mathbb{C}[S_\sigma]$ has basis element of the form χ^u for $u \in S_\sigma$ and multiplication given by $\chi^u \cdot \chi^{u'} = \chi^{u+u'}$

Now, if x is a pt in U_σ, we can consider it as an element in $\operatorname{Hom}_\mathbb{C}(\mathbb{C}[S_\sigma], \mathbb{C})$

e.g. $u = (1,1) \in \mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[x_1, x_2])$ $\quad x = (2,3)$ $\quad \chi^u(x) = x_1 \cdot x_2 = 2 \cdot 3 = 6.$

<u>Def</u>: For a cone, $x_\sigma \in U_\sigma$ is defined by

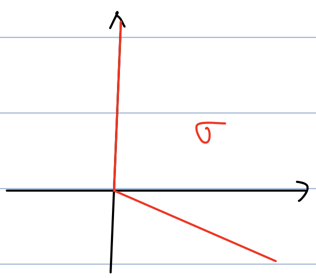
$$u \longmapsto \begin{cases} 1, & u \in \sigma^\perp \\ 0, & u \notin \sigma^\perp \end{cases} \qquad \left(\chi^u(x_\sigma) = \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0, & \text{if } u \notin \sigma^\perp \end{cases} \right.$$

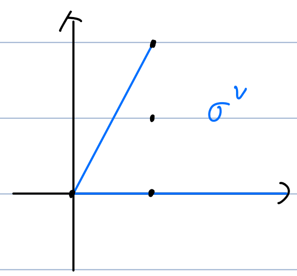
$\sigma^\perp = \{0\}$

e.g. (a) $\sigma = \mathbb{R}^n_{\geq 0} \Rightarrow x_\sigma = (0, \cdots, 0) \in \mathbb{C}^n$

(b) $\sigma' = \mathbb{R}_{\geq 0} \times \{0\} \subset \mathbb{R}^2$ $\quad U_{\sigma'} \cong \mathbb{C}_x \times \mathbb{C}^*_y$, $\sigma^\perp = \{0\} \times \mathbb{R} \Rightarrow x_\sigma = (0,1)$

(c) Let $\sigma = \operatorname{cone}((0,1), (2,-1)) \Rightarrow \sigma^\vee = \operatorname{cone}((1,0), (1,2))$

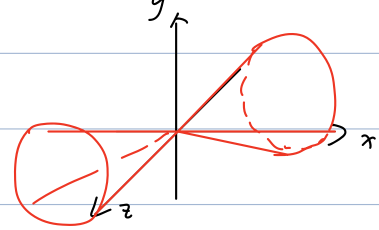




$S_\sigma = \mathbb{Z}(1,0) \oplus \mathbb{Z}(1,1) \oplus \mathbb{Z}(1,2)$

$\cong \mathbb{Z}^3$

$A_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[x, xy, xy^2] = \mathbb{C}[u,v,w]/(uw - v^2)$ $\quad U_\sigma \cong \{(x,y,z) \in \mathbb{C}^3 \mid xz = y^2\}$



$\sigma^\perp = \{0\} \Rightarrow x_\sigma = (0,0,0)$ is the only singular point of U_σ.

One can see this through either

(a) Thm: If $Y \subset \mathbb{A}^n$, which is $\mathbb{C}^n$ in our case, is an affine variety, $P \in Y$ a pt, then

Y is nonsingular at $P \iff \mathcal{O}_{P,Y}$ is regular local ring.

The check reduces to calculating the dimension of m_P/m_P^2 over $\mathbb{C}$ as a v.s.

m_P is the maximal ideal of $x_\sigma = (0,0,0) \Rightarrow m_p = (x,y,z)$ $\quad m_P^2 = (x^2, y^2, z^2, xy, xz, yz)$

$m_P/m_P^2 \cong \mathbb{C}\bar{x} \oplus \mathbb{C}\bar{y} \oplus \mathbb{C}\bar{z} \Rightarrow$ dimension $3 \neq 2 =$ dimension U_σ.

Check: U_σ is cut out by $f(x,y,z) = y^2 - xz$, $\nabla f = \langle -z, 2y, x \rangle$ vanishes at x_σ,

U_σ is singular at $(0,0,0)$.

The key reason for this to happen is giving us the following definition:

A cone is simplicial if it is the cone of a simplex / generated by linearly independent element in N, not necessarily part of a $\mathbb{Z}$-basis

$\sigma^\vee$ generated by $(1,0)$ $(1,2)$

In our case, $M_{\sigma^\vee} = \mathbb{Z} \oplus 2\mathbb{Z}$ is of index 2 in M.

Summarizing the above observation (Thm):

An affine toric variety is nonsingular $\iff$ σ is generated by part of a $\mathbb{Z}$-basis of the lattice N, where $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ for $k = \dim(\sigma)$

§ 2.2 Quotient Singularities

Continuing on the previous example, if $G = \{\pm I\} \in GL_2(\mathbb{C})$ and $\mathbb{C}^2 \cong \operatorname{Spec}(\mathbb{C}[t_1, t_2])$, then $G \curvearrowright \mathbb{C}[t_1,t_2]$ has an invariant subalgebra $\mathbb{C}[t_1,t_2]^G = \mathbb{C}[t_1^2, t_1t_2, t_2^2] \cong A_\sigma = \mathbb{C}[u,v,w]/(uw - v^2)$

$\Rightarrow U_\sigma = \mathbb{C}^2/\mathbb{Z}_2$

Generalizing this if $\sigma_m = \operatorname{Cone}((0,1), (m,-1))$ then $A_\sigma \cong \mathbb{C}[x, xy, \dots, xy^m]$

$\cong \mathbb{C}[u^m, u^{m-1}v, \dots, uv^{m-1}, v^m]$

via $x = u^m$ $\quad y = v/u$

$G = \{m\text{-th roots of unity}\}$, then $A_\sigma \cong \mathbb{C}[u,v]^G$

$\mu \in G$ acts by $\mu \cdot (u,v) = (\mu u, \mu v) \Rightarrow U_\sigma \cong \mathbb{C}^2/G$ or $\mathbb{C}^2/\mathbb{Z}_m$

To see this via toric geometry, Let N' be the lattice generated by $(0,1)$ and $(m,-1)$ in N

σ is the same cone but now considered as a cone in N', $U_{\sigma,N'} \cong \mathbb{C}^2$

Note the $N' \to N$ induces a map $U_{\sigma,N'} \to U_{\sigma,N}$

If N' is generated by me_1 & e_2, M' is generated by e_1^* and $\frac{1}{m}e_1^* + e_2^*$

corresponds to U and $V \Rightarrow A_{\sigma,N'} = \mathbb{C}[u,v]$

Now, $A_{\sigma,N} = A_{\sigma,N'} \cap \mathbb{C}[M] = A_{\sigma,N'} \cap \mathbb{C}[M']^G = A_{\sigma,N'}^G$

In general, $\overset{\text{sublattice of finite index}}{N' \subset N}$ $N'_{\mathbb{R}} = N_{\mathbb{R}}$, σ s.c.R.P.C. in $N'_{\mathbb{R}} \Leftrightarrow$ it's also s.c.R.P.C. in $N_{\mathbb{R}}$

If so, we have $V_{\sigma,N'} \longrightarrow V_{\sigma,N} \longleftarrow$ covering

$= \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M'])$ $\qquad$ $= \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$

On the level of tori, we have $0 \to N' \to N \to N/N' \to 0$, tensoring w/ the divisible grp $\mathbb{C}^*$

$$0 \to T_{N'} \to T_N \to N/N' \to 0$$

Thus, $G = N/N'$ will act on $V_{\sigma,N'}$ w/ $V_{\sigma,N} \cong V_{\sigma,N'}/G$.

Same story holds for a fan.

§2.3 One parameter subgrps & Limit points.

Goal: we want to recover fan/cone from the above 2 objects.

Def: A one-parameter subgrp of a torus T is grp homomorphism $\lambda: \mathbb{C}^* \to T$

e.g. Let $v = (b_1, \dots, b_n) \in \mathbb{Z}^n$, then $\lambda^v(t) = (t^{b_1}, \dots, t^{b_n})$

The example gives rise to all possible one-parameter subgroups of $(\mathbb{C}^*)^n$

i.e. $N \xleftrightarrow{1-1} \{\text{one-parameter subgrps of } T_N\}$

Given a character χ^u associated to $u \in M = N^\vee$,

$\chi^u \circ \lambda^v : \mathbb{C}^* \xrightarrow{\lambda^v} T_N \xrightarrow{\chi^u} \mathbb{C}^*$ sends $t \in \mathbb{C}^*$ to $t^{\langle u,v\rangle}$

The limit point of a one-parameter subgroup λ^v is $\lim\limits_{t\to 0} \lambda^v(t)$

Proposition: Let $\sigma \subseteq N_{\mathbb{R}}$ s.c.r.p.c. and $v \in N$. Then

$$v \in \sigma \iff \lim_{t\to 0} \lambda^v(t) \text{ exists in } U_\sigma$$

Moreover, if $v \in \mathrm{int}(\sigma)$, $\lim\limits_{t\to 0} \lambda^v(t) = x_\sigma$ the distinguished point.

Pf: RHS $\iff \lim\limits_{t\to 0} \chi^u \circ \lambda^v(t) = \lim\limits_{t\to 0} t^{\langle u,v\rangle}$ exists in $\mathbb{C}$ for all $u \in S_\sigma$

$\iff \langle u,v\rangle \geq 0$ for all $u \in \sigma^\vee \cap M = S_\sigma$

$\iff v \in (\sigma^\vee)^\vee = \sigma$ LHS

Def: For each $\overset{\text{cone}}{\sigma} \in \overset{\text{fan}}{\Sigma}$, the torus orbit is $O(\sigma) = T_N \cdot x_\sigma \subseteq X_\Sigma$.

Denote $N(\sigma) = N/(\sigma \cap N)$

Lemma: $O(\sigma) \simeq T_{N(\sigma)} \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$

Theorem (Orbit–cone correspondence): Let X_Σ be a toric variety associated to a fan $\Sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$

(1) $\{\text{cones } \sigma \text{ in } \Sigma\} \xleftrightarrow{1-1} \{T_N\text{-orbits in } X_\Sigma\}$

$\sigma \longleftrightarrow O(\sigma)$

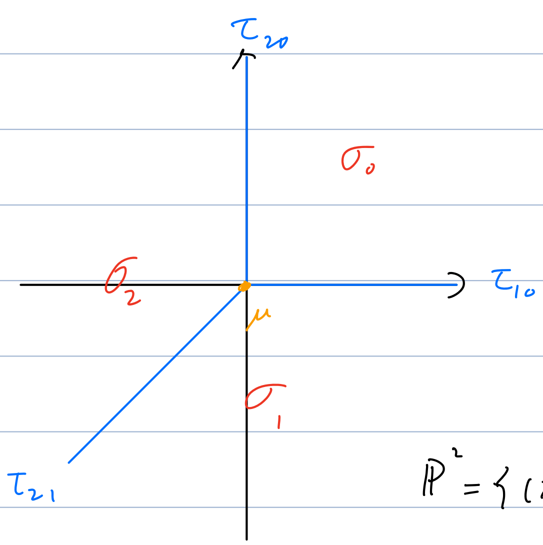
(2) $\dim O(\sigma) = n - \dim(\sigma)$

(3) $U_\sigma = \bigcup\limits_{\tau \preceq \sigma} O(\tau)$

closure is taken either in classical/Zariski topology

(4) $\tau \preceq \sigma \iff O(\sigma) \subseteq \overline{O(\tau)}$ and $\overline{O(\tau)} = \bigcup\limits_{\tau \preceq \sigma} O(\sigma)$

Ex. Let $\mathbb{P}^2 = X_\Sigma$ given by



For any $v = (a,b) \in N$, we consider

$$\lim_{t\to 0} \lambda^v(t) = \lim_{t\to 0} (1, t^a, t^b)$$

Explicit calculation gives us

$\mathbb{P}^2 = \{(x_0, x_1, x_2) \in (\mathbb{C}^3 \setminus 0)/\mathbb{C}^*\}$

$V=(a,b)$	Limit point	cone	torus orbit
$a=b=0$	$(1,1,1)$	μ	$\{x_i \neq 0 \text{ for all } i\}$
$a,b>0$	$(1,0,0)$	σ_0	$\{x_1=x_2=0,\ x_0\neq 0\}$
$a>b,\ b<0$	$(0,0,1)$	σ_1	$\{x_0=x_1=0,\ x_2\neq 0\}$
$a<0,\ a<b$	$(0,1,0)$	σ_2	$\{x_0=x_2=0,\ x_1\neq 0\}$
$a>0,\ b=0$	$(1,0,1)$	τ_{10}	$\{x_1=0,\ x_0,x_2\neq 0\}$
$a=0,\ b>0$	$(1,1,0)$	τ_{20}	$\{x_2=0,\ x_0,x_1\neq 0\}$
$a<0,\ a=b$	$(0,1,1)$	τ_{21}	$\{x_0=0,\ x_1,x_2\neq 0\}$

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§ 2.4 Compactness & Properness

Prop: Let $\varphi: N' \to N$ be morphism of lattices sending $\Sigma' \subset N'$ to $\Sigma \subset N$, the indued morphism $\varphi_*: X_{\Sigma'} \to X_\Sigma$ is proper iff $\varphi^{-1}(|\Sigma|) = |\Sigma'|$

Def: The support of a fan Σ, $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$; Σ is complete if $|\Sigma| = N_{\mathbb{R}} = \mathbb{R}^n$

Def-thm: A morphism $f: X - Y$ btw varieties is proper if for any discrete evaluation ring R w/ fractional field K, any commutative diagram has a unique lift

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{f} & X \\ \downarrow & \nearrow & \downarrow \varphi \\ \mathrm{Spec}(R) & \xrightarrow[g]{} & Y \end{array}$$

$\exists$ ord : $K \to \mathbb{Z}$ s.t. $(\mathrm{ord}|_R) \subset \mathbb{Z}_{\geq 0} \cup \{\infty\}$

Standard example : $R = k[[t]]$, $K = k((t))$

sending $f \in k((t))$ to the degree of f ?

$\mathrm{ord}(t) = 1$ $\forall f\ \exists g \in R^\times$ $f = g \cdot t^{\mathrm{ord}(f)}$

Pf $\Rightarrow$: If there's some $v' \in N'$ that's not in the cone of Σ' and $v = \varphi(v')$ in a cone of Σ, then $\varphi_*(\lambda_{v'}(z)) = \lambda_v(z)$ doesn't exists in X_Σ as $z \to 0$.

the idea of $\Leftarrow$: WLOG, $X = X_{\sigma'}$ and $Y = X_\Sigma$

The goal is to find $\sigma \in \Sigma$ s.t. $\varphi(\sigma') \subset \sigma$

f is detemined algebraically by a morphism $\alpha : S_{\sigma'} \to K$

$f^*: \mathbb{C}[S_{\sigma'}] \to K$

$\mathrm{ord}\circ\alpha : S_{\sigma'} \to \mathbb{Z}$ $\Rightarrow$ precompose φ^* gives $\underbrace{\mathrm{ord}\circ\alpha\circ\varphi^*}_{\beta}$ evaluates nonnegatively on the corresponding image of σ' $\Rightarrow \exists\, \sigma \in \Sigma$ s.t. $\beta \in (\sigma^\vee)^\vee = \sigma$

Apply the prop to $f: X_\Sigma \to \{*\}$

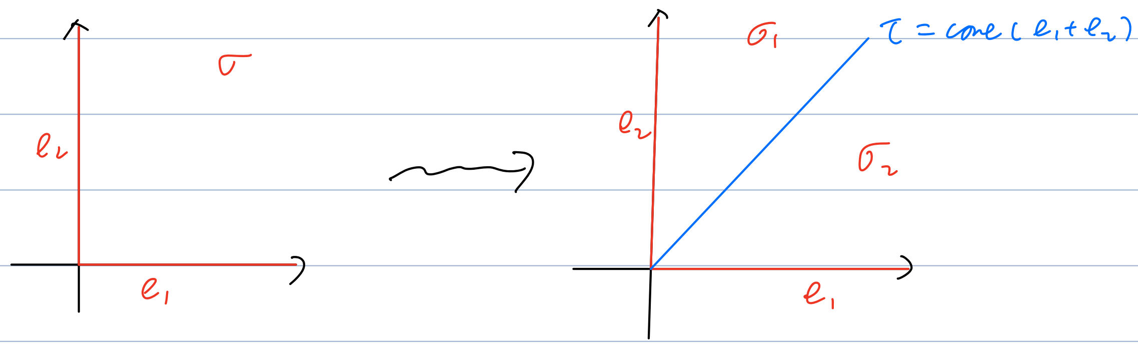
Prop: A toric variety X_Σ is cpct $\Leftrightarrow$ Σ is complete.

A fundamental example of proper map: blow up

Geometrically, the prototype blow up is given by

$$Bl_0(\mathbb{C}^n) = \{(x_i, [y_i]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid x_i y_j = x_j y_i\} \text{ w/ a cover } \underset{\mathbb{C}^n}{\underset{\wr\wr}{U_i}} = \{x_i \neq 0\}$$

In toric geometry: $U_\sigma = \mathbb{C}^2$



$$\begin{tikzcd} A_{\sigma_1} \cong \mathbb{C}[x/y, y] \cong \mathbb{C}[u,v] \arrow[dr] & & A_{\sigma_2} \cong \mathbb{C}[x, y/x] \cong \mathbb{C}[s,t] \arrow[dl] \\ & A_\tau \cong \mathbb{C}[a, b, b^{-1}] & \end{tikzcd}$$

$$\begin{tikzcd} X_{\sigma_1} & X_{\sigma_2} \\ \mathbb{C}^*_v \times \mathbb{C}^*_u \arrow[u, hook] \arrow[r, shift left] & \mathbb{C}^*_s \times \mathbb{C}^*_t \arrow[u, hook] \arrow[l, shift left] \\ (u,v) \arrow[r, mapsto] & (uv, 1/u) \\ (\tfrac{1}{t}, st) & (s,t) \arrow[l, mapsto] \end{tikzcd}$$

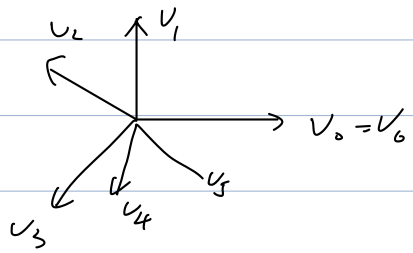
Def: Given a fan Σ, a fan Σ' refines Σ if $\forall \sigma' \in \Sigma'$ $\exists \sigma \in \Sigma$ s.t. $\sigma' \subset \sigma$ and $|\Sigma'| = |\Sigma|$

Def: Let $\sigma = \operatorname{cone}(u_1, \dots, u_r)$ smooth cone (i.e. $u_1, \dots, u_r$ is part of a basis of N). Let $u_\sigma = \sum_{i=1}^{r} u_i$ and form the cones generated by subset of $\{u_\sigma, u_1, \dots, u_r\}$ not equal to $\{u_1, \dots, u_n\}$. Denote the new fan by $\Sigma^*(\sigma)$ called the star subdivision of Σ along σ. Moreover, $X_{\Sigma^*(\sigma)} \to X_\Sigma$ is the blow up of X_Σ along τ_σ

One can generalize blow up of a fan by subdividing along any ray in the interior of a cone.

§ 2.5 Nonsingular surfaces.

2-dimensional nonsingular complete toric varieties are characterized by the fan associated to a sequence of lattice pts $v_0, v_1, \dots, v_{d-1}, v_d = v_0$ ordered counterclockwisely s.t. any 2 adjacent elements form a basis for N.



Fact:

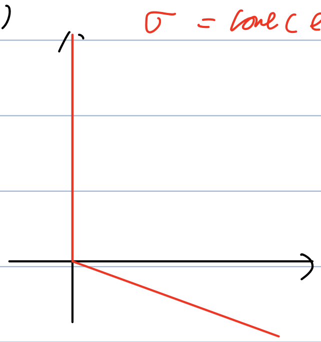
$d=3$	$X_3 = \mathbb{P}^2$	
$d=4$	$X_4 = \mathbb{F}_a$	Hirzebruch surfaces
$d \geq 5$	blows up of above.	

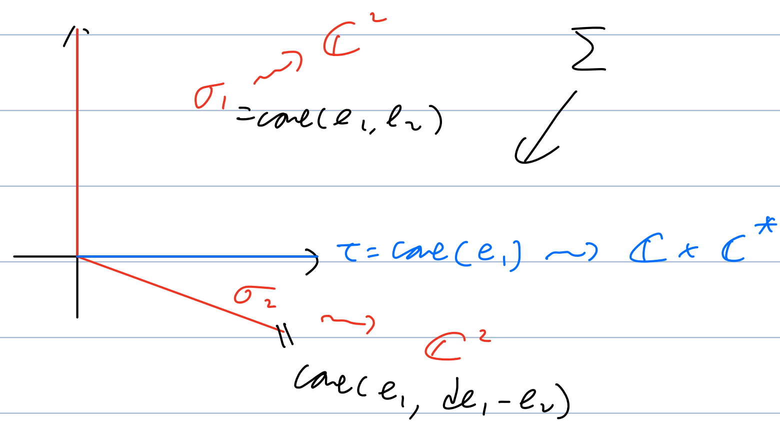
2.6. Resolution of singularities

<u>Def</u>: A proper morphism $\varphi : Y \to X$ is a resolution of singularities of X if Y is a smooth variety and φ induces $Y \setminus \varphi^{-1}(X_{sing}) \cong X \setminus X_{sing}$

Idea: We keep blowing up by refining the fan until all cones are smooth.

<u>Ex</u>: (a) $\sigma = \mathrm{cone}(e_2, de_1 - e_2)$

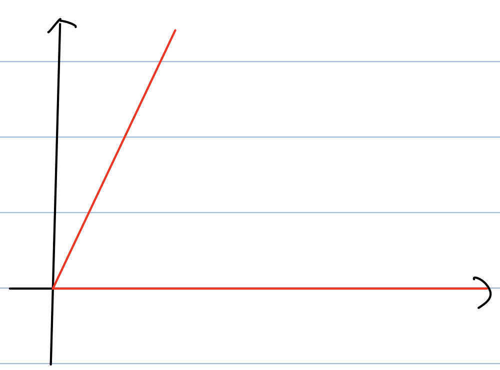


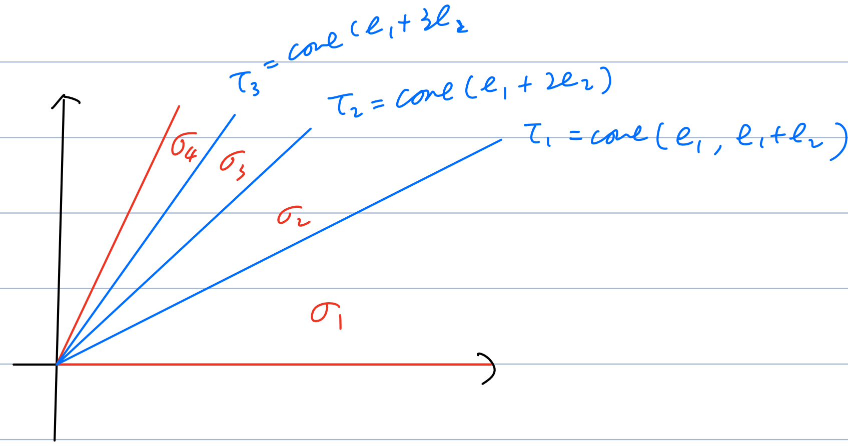


$(X_\sigma)_{sing} = \{x_\sigma\}$

In particular, $\varphi^{-1}(\{x_\sigma\})$ is the closure of the T_N-orbit $O(\tau)$, a curve $E \simeq \mathbb{P}^1$ called exceptional divisor on X_Σ

(b) $\sigma = \mathrm{cone}(e_1, e_1 + 4e_2)$





σ_i is smooth & $X_{\sigma_i} \cong \mathbb{C}^2 \Rightarrow X_\Sigma \xrightarrow{\varphi} X_\sigma$ and $X_\Sigma \setminus \varphi^{-1}(\{x_\sigma\}) = X_\sigma \setminus \{x_\sigma\}$

w/ $\varphi^{-1}(\{x_\sigma\}) = E = \bar{O}(\tau_1) \cup \bar{O}(\tau_2) \cup \bar{O}(\tau_3)$

$\bar{O}(\tau_1) \pitchfork \bar{O}_{\tau_2}(\tau_2)$ at x_{σ_2} & $\bar{O}(\tau_2) \pitchfork \bar{O}(\tau_3)$ at x_{σ_3}

Thm: Let X_Σ be normal toric variety. $\exists$ a smooth fan Σ' refines Σ and $\varphi : X_{\Sigma'} \to X_\Sigma$ is a resolution of singularities

Continued Fraction:

Prop: For a 2-dim s.c.r.p.c. $\sigma \subset N_{\mathbb{R}} = \mathbb{R}^2$, $\exists$ a basis e_1, e_2 for N s.t.
$\sigma = \text{cone}(e_2, de_1 - ke_2)$ where $d > 0$, $0 \le k < d$, $\gcd(d,k) = 1$.

Pf: (Fact: given $l, d \in \mathbb{Z}_{>0}$, $\exists!\, s, k$ such that $l = sd - k$ w/ $0 \le k < d$) (*)

Let $\sigma = \text{cone}(u_1, u_2)$ for primitive $u_1, u_2 \Rightarrow \exists\, e_1'$ s.t. e_1' $e_2 = u_1$ s.t. form a basis $\Rightarrow u_2 = de_1' + le_2$ for $d \neq 0$, WLOG we assume $d > 0$.

By (*), $\exists\, s, k$ s.t. $l = sd - k$ for $0 \le k < d$. Let $e_1 = e_1' + se_2$ then e_1, e_2 form a basis and $u_2 = de_1 + (l - sd)e_2 = de_1 - ke_2$ $\gcd(d,k) = 1$ b/c u_2 is primitive.

This prop helps us to transform the cone we want to blow up into this canonical form.

Resolution of Sing.: step 0: write σ $\left(= \text{cone}(e_2, de_1 - ke_2)\right)$ in this form

step 1: refine σ by adding e_1

$\sigma_0 = \text{cone}(e_1, e_2)$ $\quad \sigma_1 = \text{cone}(e_1, de_1 - ke_2)$

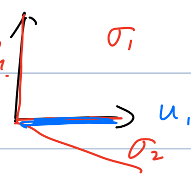
Next, run the prop for σ_1

Let $u_0 = e_1$, $u_1 = a_1 e_1 - e_2$ for $d = a_1 k - k_1$ w/ $a_1 \ge 2$, $0 < k_1 \le k$ by (*)

$\Rightarrow de_1 - ke_2 = (a_1 k - k_1) e_1 - ke_2 = k(a_1 e_1 - e_2) - k_1 e_1 = ku_1 - k_1 u_0$

$\Rightarrow \sigma_1' = (u_0, u_1)$ and $\sigma_2 = \text{cone}(u_1, ku_1 - k_1 u_0)$

Recursive Step: do this again and again until all cones are smooth.



$d = a_1 \underset{\sim}{k} - k_1$

$k = a_2 \underset{\sim}{k_1} - k_2$

$\cdots$

$\cdots$

$$\frac{d}{k} = a_1 - \frac{k_1}{k}$$

$$= a_1 - \cfrac{1}{a_2 - \cfrac{k_2}{k_1}}$$

$$\cdots$$

$$= a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_r}}}$$

called Hirzebruch–Jung continued fraction.