Surgery Constructions using ATF

1. Singularity of type $\frac{1}{n}(1,a)$

On $\mathbb{C}^2$ we have the $\mathbb{Z}_n$-action

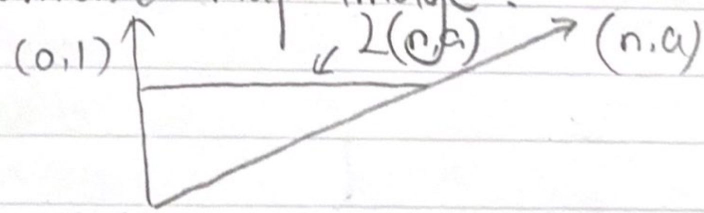
$$\mu \cdot (z, w) = (\mu z, \mu^a w)$$

in general, it's an orbifold w/ singularity at the origin, called an $\frac{1}{n}(1,a)$ singularity.

Hamiltonian T^2-action on $\mathbb{C}^2/\mathbb{Z}_n$:

$$\mu(z,w) = \left(\tfrac{1}{2}|w|^2,\ \tfrac{1}{n}\left(\tfrac{1}{2}|z|^2 + \tfrac{a}{2}|w|^2\right)\right)$$

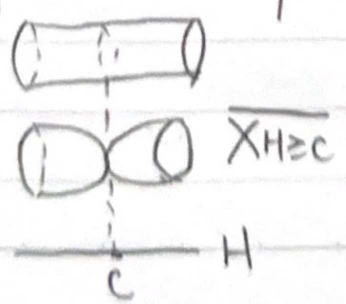
- moment map image:



- non-Delzant corner at the origin
- has $\ell(n,a)$ inside it.
- the main testing ground of surgeries in this presentation.

2. Symplectic Cut

Prototypical Example: $(\mathbb{R}_p \times S^1_q,\ dp \wedge dq,\ H = p)$



Goal: reduce a level set while keeping others.

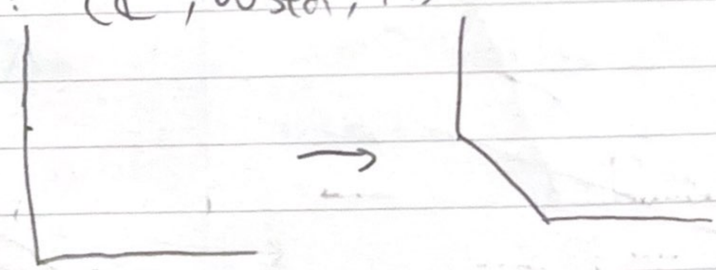
Def: Let (X,ω,H) w/ ϕ_H^t a circle action.
Pick c a regular value, consider $(X\times\mathbb{C},$
$\omega + dp\wedge dq$, $\tilde{H}_c(x,\xi) = H(x) - c - \frac{1}{2}|\xi|^2)$.
$\overline{X_{H\geq c}} := \tilde{H}_c^{-1}(0)//S^1$.

Why this meets our goal?
$\tilde{H}_c^{-1}(0) = \{(p,\xi) \mid H(p) = c + \frac{1}{2}|\xi|^2\}$

$|\xi| \neq 0$, not identifying pts in X

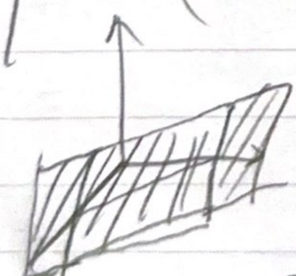
$|\xi| = 0$, reducing $\{H(p)=c\}$.

Example: $(\mathbb{C}^2, \omega_{std}, H)$, $H = \Sigma H_1 + H_2$



$(\mathbb{C}^2, \omega_{std}) \mapsto (\mathcal{O}(-1), \omega)$

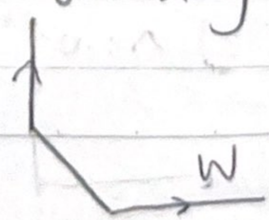
Why? ① $(\mathbb{C}^2\times\mathbb{C}, H_1, H_2, H_3)$ is toric,



② $\tilde{H}_c = H_1 + H_2 - H_3 + 1$ cuts out this plane

③ proj to x-y plane

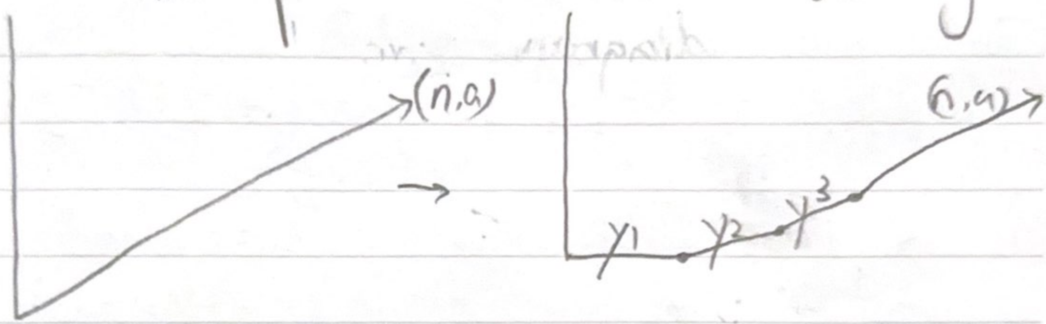
$\rightsquigarrow$



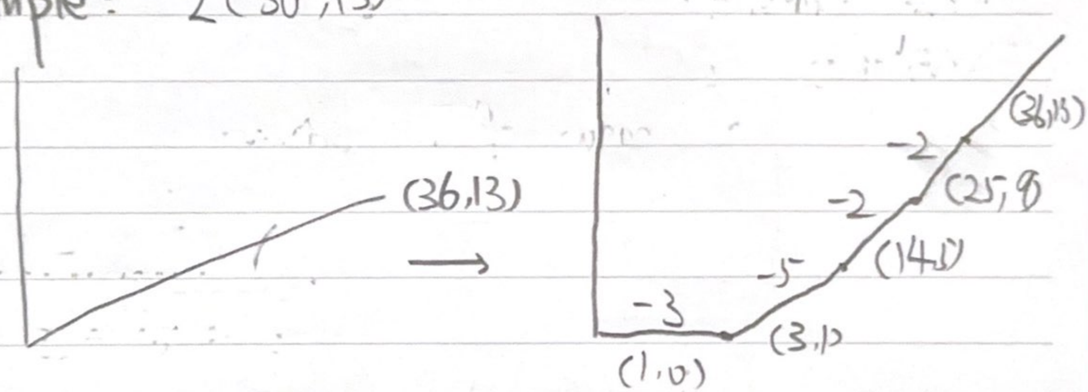
- self-intersection $V \wedge W$
- symplectic area is affine length $\cdot 2\pi$

3. Resolve $\frac{1}{n}(1,a)$.

Can cut multiple times to resolve the sing.



Example: $L(36,13)$



- $\frac{36}{13} = 3 - \cfrac{1}{5 - \cfrac{1}{2 - \frac{1}{2}}} = 3 - \cfrac{1}{5 - \frac{2}{3}} = 3 - \frac{3}{13} = \frac{36}{13}$

same holds for any (n,a)

- called the minimal resolution of $\frac{1}{n}(1,a)$
- symplectic filling of $L(n,a)$
- $\beta_2 = 4$

Goal: Do surgeries to make β_2 smaller.

4. Rational Ball $B_{1,p,q}$

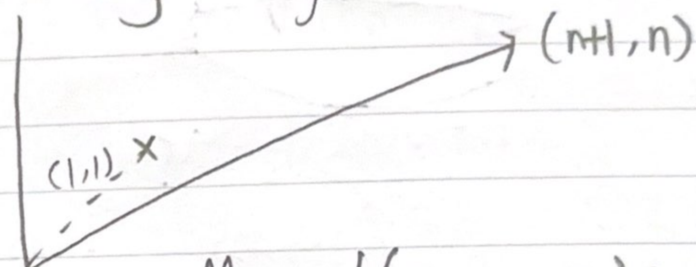
Consider the Milnor fiber

$$M_p = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + z_3 + t = 0\}$$

$H(z_1, z_2, z_3) = (|z_3|^2, \frac{1}{2}(|z_1|^2 - |z_2|^2))$

defines a Hamiltonian $\mathbb{R} \times S^1$ action.

ATF base diagram from H:

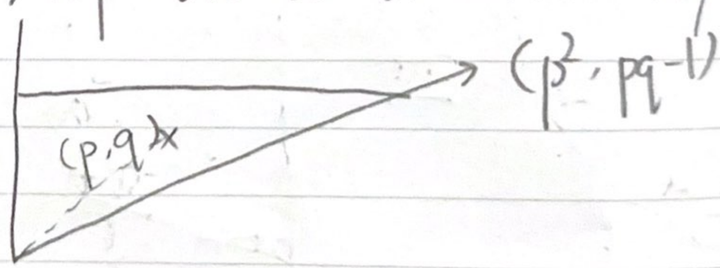


Now consider $M_p = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 + z_3^p + t = 0\}$.

- p nodal focus-focus critical pts in the critical fiber
- $\mathbb{Z}_p$ acts by $\mu \cdot (z_1, z_2, z_3) = (\mu z_1, \mu^{-1} z_2, \mu^q z_3)$ permutes the focus-focus critical pts.

$\rightsquigarrow M_p/\mathbb{Z}_p$ admits an ATF w/

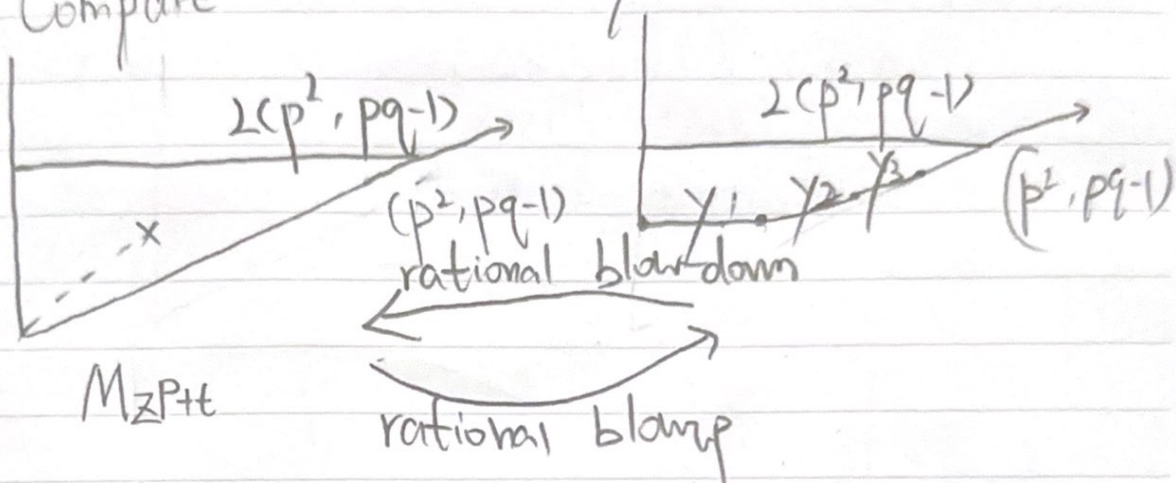
$B_{1,p,q}$



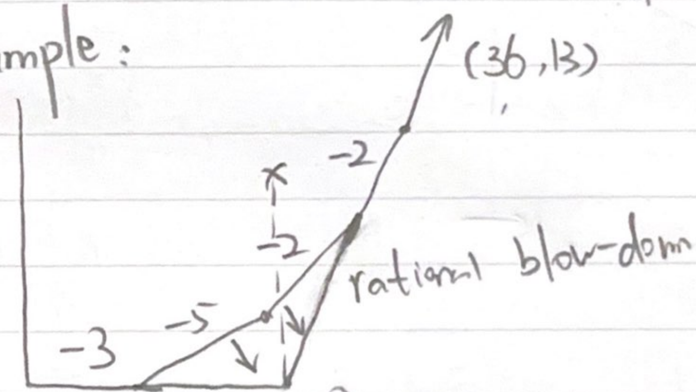
- $B_{1,p,q}$ deforms to $L_{p,q} \rightsquigarrow B_{1,p,q}$ rational
- Contains $L(p^2, pq-1)$!

5. Rational Blow-down/up

Compare



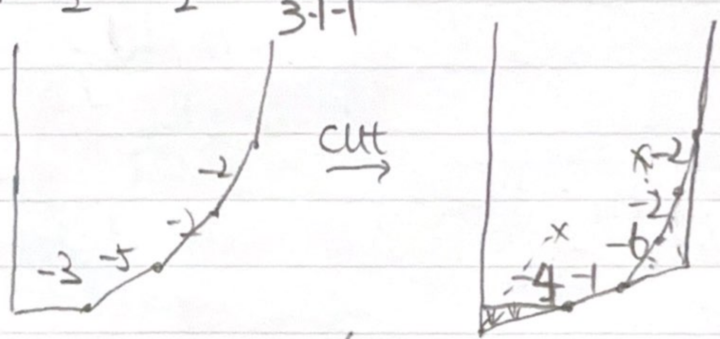
Example:



· B_2 4→1

· another filling of $L(36,13)$

$$5-\frac{1}{2}=\frac{9}{2}=\frac{3^2}{3\cdot1-1}$$



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