1. Markov Triples

Definition 1. *The Diophantine equation*

$$a^2 + b^2 + c^2 = 3abc$$

is called the **Markov equation***.*

The Markov equation arises in a number of fields [1]. In particular, we concern ourselves with positive integer solutions to the Markov equation, known as *Markov triples*.

Lemma 2. *If* (a, b, c) *is a Markov triple, then so is* $(a, b, 3ab - c)$.

Proof. Suppose (a, b, c) is a Markov triple. Then observe

$$\begin{aligned} a^2 + b^2 + c^2 &= 3abc \\ a^2 + b^2 + c^2 + 9a^2b^2 - 6abc &= 3abc + 9a^2b^2 - 6abc \\ a^2 + b^2 + (3ab - c)^2 &= 3ab(3ab - c). \end{aligned}$$

We know $3ab-c$ is an integer from its construction of integers a, b, c, and further is positive as the left hand size is the sum of two positive integers and one non-negative integer. Thus $(a, b, 3ab - c)$ is also a Markov triple. □

Definition 3. *The operation of replacing a Markov triple* (a, b, c) *with* $(a, b, 3ab - c)$ *is called a* **mutation** *on* c*.*

Given certain properties of a, b, c that form a Markov triple, we can describe properties its mutation on c will have.

Lemma 4. *If* (a, b, c) *is a Markov triple such that* $a \leq b \leq c$*, then* b *lies in the closed interval between* c *and* $3ab-c$*. If* $a \lneq b$*, then* b *lies in the interior of this interval. If* $b \lneq c$*, then* $3ab-c \leq b$*.*

Proof. The Markov triple (a, b, c) has associated quadratic function $f(x) = x^2 - 3abx + a^2 + b^2$ with positive integer roots c and $3ab - c$. Observe

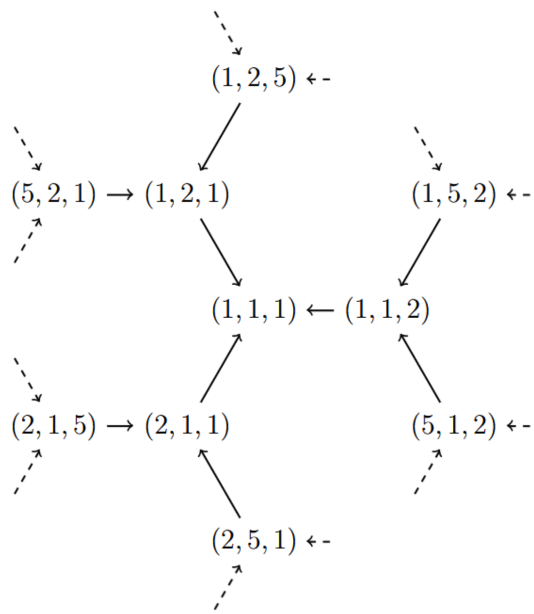
$$f(b) = b^2 - 3ab^2 + a^2 + b^2 = (2 - 3a)b^2 + a^2 \leq a^2 - b^2 \leq 0.$$

Hence b lies between the roots of f, and is strictly between the roots if $a^2 - b^2 < 0$. □

Lemma 5. *The only Markov triple with no unique largest element is* $(1, 1, 1)$.

Proof. Suppose (a, b, c) is a Markov triple with $a \leq b = c$. Then substituting $b = c$ in the Markov equation gives $a^2 + 2b^2 = 3ab^2$; that is, $a^2 = (3a - 2)b^2$. By definition $a \geq 1$, so $3a - 2 \geq 1$ and thus $a^2 \geq b^2$. So $a = b = c$, and substituting in the Markov equation grants $a^2 = (3a-2)a^2$. Hence $3a - 2 = 1$ implies that $a = b = c = 1$. □

With these properties in mind, we can construct a graph in which vertices are Markov triples and edges connect vertices related by a mutation. This graph is a connected tree which we will call the **Markov tree**. A small portion of the Markov tree can be seen below, as shown in [2].



Theorem 6. *The Markov tree is connected.*

Proof. Let (a, b, c) be a Markov triple with $c > a, b$. After performing a mutation on c, by Lemma 4 this decreases the value of the largest element. If there is still a unique largest element, perform another mutation. Repeat this procedure until no unique largest element exists. The process is guaranteed to terminate as a, b, c must remain strictly positive. By Lemma 5, we know $(1, 1, 1)$ is the unique Markov triple with a repeated largest element. Thus in the Markov tree we have a path from any Markov triple to the root vertex $(1, 1, 1)$, and the Markov tree is hence connected. □

Lemma 7. *We can define a global choice of orientation on the edge of the Markov tree in the following manner: If an edge connects two Markov triples (a, b, c) and $(a, b, 3ab - c)$, then the edge is directed towards the Markov triple with the smaller maximal element.*

Proof. To confirm this orientation is well-defined, we need to show maximal elements in (a, b, c) and $(a, b, 3ab - c)$ are different. WLOG, suppose $a \leq b$ and $c < 3ab - c$. By Lemma 4, $c \leq b \leq 3ab - c$. Hence $\max(a, b, c) = b$ and $\max(a, b, 3ab - c) = 3ab - c$. It remains to see $b \neq 3ab - c$, which is only possible if $a = b$ by Lemma 5, in which $a = b = c = 1$ and orientation around this root vertex is clear. □

Theorem 8. *The Markov tree is a tree.*

Proof. By Theorem 6, we have already seen that the Markov tree is connected. It remains to show that the Markov tree is a tree; that is, there are no cycles. In particular, there is a unique path from any vertex directed downwards to the root vertex $(1, 1, 1)$. Say the distance from a vertex to the root vertex is the height of the path.

Consider two vertices t_1, t_2. Following the paths down to $(1,1,1)$, there must exist an intersection vertex m. Then we have a simple path P from t_1 to m to t_2. Suppose there exists another simple path Q which also connects t_1 to m to t_2. Then Q and P must differ by at least an edge. Let e_i be an edge in Q that is not in P which maximizes height. There are three possibilities:

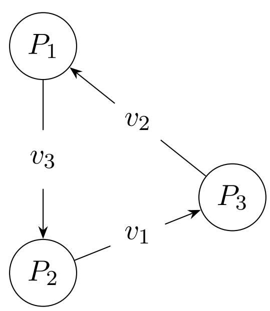
(1) ($e_i = e_1$) Since e_i is not in P, it must be an upward path from t_1. Since e_i is a highest edge, the next edge must return along the complement of e_i as the only downward path. This contradicts Q as a simple path.
(2) (e_i is the last edge) We arrive at the same contradiction as the previous case.
(3) (Otherwise) The edge e_i has adjacent edges e_{i-1}, e_{i+1} in the path. At least one of these must start at the highest point of e_i. Since it cannot go higher, it must be the reverse of e_i, contradicting Q as a simple path.

Hence there is a unique simiple path between any two vertices and the Markov tree is indeed a tree. □

2. Vianna Triangles

Definition 9. *A* **Vianna triangle** *is an almost toric diagram whose edges are v_1, v_2, v_3 with affine lengths ℓ_1, ℓ_2, ℓ_3 and whose vertices P_1, P_2, P_3 are modelled on the T-singularities $\frac{1}{d_k p_k^2}(1, d_k p_k q_k - 1)$ for $k = 1, 2, 3$.*

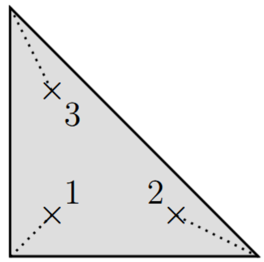
We call d_k, p_k, q_k, ℓ_k for $k = 1, 2, 3$ the **Vianna data** of a Vianna triangle. For example, we can view a Vianna triangle $D(P_1, P_2, P_3)$ below:



Theorem 10 (Vianna [3])**.** *For every Markov triple p_1, p_2, p_3, there is a Vianna triangle $D(p_1, p_2, p_3)$ with the following properties:*

(1) The diagram $D(1,1,1)$ is obtained from the standard toric diagram of $\mathbb{CP}^2$ by performing three nodal trades.
(2) The diagram $D(p_1, p_2, p_3)$ is a triangle with three base-nodes n_1, n_2, n_3, obtained by iterated mutation on $D(1,1,1)$ (in particular, the associated almost toric manifold is $\mathbb{CP}^2$).
(3) For $k = 1, 2, 3$, there is an integer q_k and a Lagrangian pinwheel of type (p_k, q_k) living over the branch cut which connects n_k to a corner P_k.
(4) The affine length of the edge opposite the corner P_k is $3p_k/(p_{k+1}p_{k+2})$ where indices are taken modulo 3.

The Vianna triangle $D(1,1,1)$ with data $d_1 = d_2 = d_3 = 1$, $p_1 = p_2 = p_3 = 1$, and $\ell_1 = \ell_2 = \ell_3 = 3$ is given below, as shown in [2]:



Now, we collect a few lemmas to prove Vianna's Theorem.

Lemma 11. *If $\vec{v}_k$ denotes the primitive integer vector along v_k, then we have the relation*

$$\vec{v}_k \wedge \vec{v}_{k+1} = d_{k+2}p_{k+2}^2.$$

Corollary 12. *Thus,*

$$\ell_1\ell_2 d_3 p_3^2 = \ell_2\ell_3 d_1 p_1^2 = \ell_3\ell_1 d_2 p_2^2.$$

For ease of notation, let K be defined as the value in Corollary 12, and let the total affine length $\ell_1 + \ell_2 + \ell_3$ be given by L. Now, we can show that L and K are unchanged by mutation to help prove Vianna's Theorem.

Corollary 13. *We have* $\ell_k = \dfrac{p_k}{p_{k+1}p_{k+2}}\sqrt{\dfrac{Kd_k}{d_{k+1}d_{k+2}}}$.

Proof. WLOG, set $k = 3$. By Corollary 12,

$$\ell_1 = \frac{K}{\ell_3 d_2 p_2^2}, \quad \ell_2 = \frac{K}{\ell_3 d_1 p_1^2}, \quad \ell_1\ell_2 d_3 p_3^2 = K$$

so by substitution

$$\frac{K^2 d_3 p_3^2}{\ell_3^2 d_1 d_2 p_1^2 p_2^2} = K \quad \Longrightarrow \quad \ell_3^2 = \left(\frac{p_3^2}{p_1^2 p_2^2}\right)\left(\frac{Kd_3}{d_1 d_2}\right).$$

Thus $\ell_3 = \dfrac{p_3}{p_1 p_2}\sqrt{\dfrac{Kd_3}{d_1 d_2}}$ as desired. □

Corollary 14. *We have*

$$d_1p_1^2 + d_2p_2^2 + d_3p_3^2 = \frac{L\sqrt{d_1d_2d_3}}{\sqrt{K}}p_1p_2p_3.$$

Lemma 15. *The eigenline at vertex P_{k+2} points in the direction* $\dfrac{\vec{v}_{k+1} - \vec{v}_k}{d_{k+2}p_{k+2}}$.

Proof. Making an integral affine transformation, we can assume that $\vec{v}_k = (0,-1)$ and $\vec{v}_{k+1} = (d_{k+2}p_{k+2}^2, d_{k+2}p_{k+2}q_{k+2} - 1)$. In these coordinates, the eigenline points in the (p_{k+2}, q_{k+2}) direction, which is $\frac{\vec{v}_{k+1} - \vec{v}_k}{d_{k+2}p_{k+2}}$ as desired. □

Lemma 16. *If we perform a mutation on the vertex P_3 then we obtain a new Vianna triangle with data:*

$$d_1' = d_1, \qquad d_2' = d_2, \qquad d_3' = d_3$$

$$p_1' = p_1, \qquad p_2' = p_2, \qquad p_3' = \frac{dp_1^2 + d_2p_2^2}{d_3p_3}$$

$$\ell_1' = \frac{\ell_3 d_2 p_2^2}{d_1p_1^2 + d_2p_2^2}, \qquad \ell_2' = \frac{\ell_3 d_1 p_1^2}{d_1p_1^2 + d_2p_2^2}, \qquad \ell_3' = \ell_1 + \ell_2.$$

Corollary 17. *The values K and L are unchanged by mutation at any vertex P_k.*

Proof. (Vianna) The triangle $D(1,1,1)$ is a Vianna triangle with Vianna data $d_1 = d_2 = d_3 = 1$, $p_1 = p_2 = p_3 = 1$, and $\ell_1 = \ell_2 = \ell_3 = 3$. In this case, $K = L = 9$. By Corollaries 14 and 17, performing iterated mutations on $D(1,1,1)$ will produce new Vianna triangles with Vianna data d_k, p_k, ℓ_k such that $p_1^2 + p_2^2 + p_3^3 = 3p_1p_2p_3$ and $\ell_k = 3p_k/(p_{k+1}p_{k+2})$.

Let $D(p_1, p_2, p_3)$ be the triangle associated with the Markov triple (p_1, p_2, p_3). Mutation at p_3 gives a new Markov triple $p_1, p_2, p_3' = 3p_1p_2 - p_3$. These are the only two Markov triples containing p_1, p_2. ☐

Open Question 18. *Characterize which quadrilaterals arise as mutations of a square or a rectangle.*

References

[1] Martin Aigner. Markov's Theorem and 100 Years of the Uniqueness Conjecture: A Mathematical Journey from Irrational Numbers to Perfect Matchings. Springer Cham, 2013.

[2] Jonathan David Evans. Lectures on lagrangian torus fibrations, 2022.

[3] R. Vianna. Infinitely many exotic monotone lagrangian tori in $\mathbb{CP}^2$. J. Topol., 9(2):535–551, 2016.