Riemannian Manifolds with Integrable Geodesic Flows

Mohammad Tinati

1 Introduction

These notes focus on the interplay between Riemannian geometry and Hamiltonian dynamics, with a special emphasis on the integrability of geodesic flows. A central question is: under what conditions does a Riemannian manifold admit a metric whose geodesic flow is integrable in the sense of Hamiltonian systems?

To place this question in a precise mathematical setting, we begin by framing the geodesic flow of a Riemannian manifold as a Hamiltonian system. This perspective emerges naturally once we interpret the tangent or cotangent bundle of the manifold as a symplectic manifold. Within this framework, a geodesic flow can be viewed as the flow of a Hamiltonian vector field generated by a suitable "energy function." Understanding integrability then becomes tantamount to finding a complete set of integrals in involution for this Hamiltonian system.

This note is structured in three parts:

1. Symplectic structure of the cotangent bundle of a Riemannian manifold
2. Classical examples of integrable Hamiltonian systems
3. Thimm's method and Duran's theorem

In this initial portion, we focus on the foundational aspects: how a Riemannian metric induces a natural Hamiltonian on the cotangent bundle, and how the canonical symplectic structure on T^*M allows us to reinterpret geodesics as Hamiltonian flows.

2 Symplectic Preliminaries

2.1 Symplectic Manifolds and Hamiltonian Dynamics

A *symplectic manifold* (M, ω) is an even-dimensional manifold M endowed with a closed, nondegenerate 2-form ω. The nondegeneracy condition ensures that at each point $x \in M$, the form

$$\omega_x : T_xM \times T_xM \to \mathbb{R}$$

is a symplectic bilinear form.

Given a smooth Hamiltonian function $H : M \to \mathbb{R}$ on a symplectic manifold (M, ω), we define the *Hamiltonian vector field* ξ_H uniquely by the relation

$$\omega(\xi_H, \cdot) = dH(\cdot).$$

The flow generated by ξ_H is called the *Hamiltonian flow* of H. By construction, H is an integral of its own flow. Moreover, Hamiltonian flows preserve the symplectic form: if g^t is the flow of ξ_H, then

$$(g^t)^*\omega = \omega.$$

2.2 Poisson Brackets and Integrals in Involution

On a symplectic manifold, the space $C^\infty(M, \mathbb{R})$ of smooth functions inherits the structure of a Poisson algebra. The *Poisson bracket* of two functions F, G is defined by

$$\{F, G\} = \omega(\xi_F, \xi_G) = dF(\xi_G).$$

This bracket is bilinear, skew-symmetric, and satisfies the Jacobi identity, making $(C^\infty(M), \{\cdot, \cdot\})$ into a Lie algebra.

If the Hamiltonian H admits $n = \frac{1}{2}\dim(M)$ functionally independent integrals $F_1 = H, F_2, \ldots, F_n$ in *involution*, i.e. $\{F_i, F_j\} = 0$ for all i, j, then the Hamiltonian system is said to be *integrable*. By Liouville's theorem, such integrable systems have solutions that lie on invariant tori. The functions $F_1 = H, F_2, \ldots, F_n$ are independent if for an open dense subset of M, we have

$$dF_1 \wedge \cdots \wedge dF_n \neq 0.$$

3 Riemannian Geometry and the Geodesic Flow

3.1 From Geodesics to Hamiltonian Flows

Let (M, g) be a Riemannian manifold. Geodesics are curves that locally minimize distance and can be characterized by the *geodesic equation*, derived from the Levi-Civita connection associated with g. Each geodesic $\gamma(t)$ with initial data $(x, v) \in TM$ defines a flow g^t on the tangent bundle: given (x, v), evolve it along the unique geodesic determined by v at x. This *geodesic flow* acts as

$$g^t(x, v) = (c(t), \dot{c}(t)),$$

where $c(t)$ is the geodesic with initial conditions $c(0) = x$ and $\dot{c}(0) = v$.

To link geodesic flow with Hamiltonian dynamics, we move to the cotangent bundle T^*M. Through the metric g, we can identify TM and T^*M. Consider the *musical isomorphism*:

$$v \in T_xM \mapsto g_x(v, \cdot) \in T_x^*M.$$

Using this identification, every vector $v \in T_xM$ corresponds to a covector $p \in T_x^*M$, and we define the Hamiltonian

$$H(x, p) = \tfrac{1}{2} g^{ij}(x) p_i p_j,$$

where (g^{ij}) is the inverse of the metric tensor (g_{ij}). This Hamiltonian $H : T^*M \to \mathbb{R}$ represents the kinetic energy of a free particle moving on M.

3.2 The Canonical Symplectic Structure on T^*M

The cotangent bundle T^*M admits a canonical symplectic form. In local coordinates (x^i, p_i) on T^*M, define the 1-form

$$\theta = \sum_{i=1}^{n} p_i dx^i.$$

Then the canonical symplectic form on T^*M is

$$\omega = -d\theta = \sum_{i=1}^{n} dx^i \wedge dp_i.$$

This form is exact, nondegenerate, and closed, making (T^*M, ω) a symplectic manifold.

Since $H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j$ is a smooth function on T^*M, it generates a Hamiltonian flow via the vector field ξ_H defined by

$$\omega(\xi_H, \cdot) = dH(\cdot).$$

This Hamiltonian flow is precisely the *cogeodesic flow*, and its projection onto M yields the original geodesic flow. Thus, the geodesic flow on a Riemannian manifold can be realized as the Hamiltonian flow of a natural mechanical system on (T^*M, ω).

4 Geometric Interpretations and Horizontal/Vertical Decompositions

Another viewpoint is to consider the tangent bundle TM itself. Using the Levi-Civita connection ∇, one can decompose $T(TM)$ into vertical and horizontal subbundles. The vertical space at a point $\theta \in TM$ consists of all vectors tangent to the fiber, while the horizontal space is given by the kernel of the connection map.

A suitable 2-form ω defined on TM using these horizontal and vertical projections can also yield a symplectic structure. From this perspective, the geodesic flow again emerges as a Hamiltonian flow, but now directly on (TM, ω) rather than passing through T^*M. This approach offers insights into the geometric nature of the system, linking curvature and geometric invariants to dynamical properties.

5 Classical Examples on Two-Dimensional Surfaces

A principal class of examples where one finds integrable geodesic flows is that of two-dimensional surfaces. Two of the most classical examples are:

1. The two-dimensional sphere $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ of constant curvature.

2. The flat torus.

On S^2, geodesics are great circles, i.e., intersections of S^2 with planes through the origin. Since all geodesics are closed, one finds more integrals than just the dimension count suggests. Instead of having only two independent first integrals, there are three independent first integrals. One can think of these integrals as corresponding to infinitesimal rotations of the sphere.

For example, consider the integral corresponding to rotation about the x_3-axis:

$$f_3(x,p) = p(\xi_3(x)),$$

where $p \in T_x^* S^2$, and $\xi_3 = \partial/\partial\varphi$ is the vector field associated to the standard spherical coordinate φ on S^2. In Cartesian coordinates,

$$\xi_3(x_1, x_2, x_3) = (-x_2, x_1, 0).$$

In terms of the tangent bundle, these integrals have a particularly transparent geometric interpretation. If we represent a geodesic by $(x, \dot{x}) \in TS^2$, the three integrals form a vector integral

$$F(x, \dot{x}) = (f_1, f_2, f_3) = [x, \dot{x}],$$

where we view $x \in S^2 \subset \mathbb{R}^3$ and $\dot{x} \in T_x S^2$ as vectors in $\mathbb{R}^3$. The vector F is orthogonal to the plane in which the geodesic lies.

Among these three integrals, any two commute, and together with the Hamiltonian

$$H = \tfrac{1}{2}(f_1^2 + f_2^2 + f_3^2),$$

they provide an integrable structure on S^2.

On the flat torus with metric $ds^2 = dx_1^2 + dx_2^2$ (with x_1, x_2 taken modulo 2π), the geodesics are straightforward: $x_i(t) = c_i t$. These geodesics wrap around the torus, generally in a quasi-periodic fashion. The commuting first integrals in this case are simply the momenta (p_1, p_2). The Hamiltonian of the flow on the cotangent bundle is

$$H(x_1, x_2, p_1, p_2) = \tfrac{1}{2}(p_1^2 + p_2^2),$$

or, in a more general form,

$$H(x_1, x_2, p_1, p_2) = \tfrac{1}{2}(a p_1^2 + 2b p_1 p_2 + c p_2^2),$$

and these momenta provide integrability of the geodesic flow on the flat torus.

6 Metrics of Revolution and Clairaut's Theorem

Another important example of integrable geodesic flow arises on surfaces of revolution. Let us consider a surface of revolution about the x_3-axis. The classical Clairaut's theorem states:

Theorem 1 (Clairaut)**.** *The geodesic flow on a surface of revolution admits a non-trivial linear integral and, therefore, is integrable.*

Geometrically, if we let ψ denote the angle between the geodesic and the parallel (circle of revolution) on the surface, and r the distance from the revolution axis, then the Clairaut integral takes the simple form

$$r \cos\psi.$$

This integral is directly analogous to the previously mentioned integral f_3 for the sphere S^2.

7 Liouville Metrics and Quadratic Integrals

The famous Liouville theorem provides another classical example of an integrable geodesic flow. Consider a metric of the form

$$ds^2 = (f(x_1) + g(x_2))(dx_1^2 + dx_2^2). \tag{1}$$

Liouville showed that the corresponding geodesic flow admits a non-trivial *quadratic* integral of the form

$$F(x, p) = \frac{g(x_2)p_1^2 - f(x_1)p_2^2}{f(x_1) + g(x_2)}. \tag{2}$$

As a consequence, the geodesic flow is integrable.

This construction is generally local in nature. One can often find local coordinates (x_1, x_2) on a two-dimensional surface to put the metric in the Liouville form (1), and thus locally obtain the integral (2). However, when attempting to define such an integral globally on the entire surface, one may encounter topological obstructions. In particular, while locally integrable structures exist, global integrability may fail.

A notable illustration arises in the case of a surface with constant negative curvature and genus $g > 1$. Locally, the geodesic flow may admit a quadratic integral (obtained by combining several linear integrals), but no such integral can be defined globally. Hence, the integrability may hold only in a local sense, underscoring the subtleties involved in translating local integrability into a global condition.

8 Thimm's Method

In earlier parts, we have seen how integrable geodesic flows can arise from classical examples, such as surfaces of revolution or symmetric spaces. We now turn our attention to a powerful and more general construction due to Thimm [17], which uses group actions and moment maps to generate integrable Hamiltonian systems. This method effectively builds non-trivial first integrals in involution from invariants on a Lie algebra chain, providing a systematic way to prove complete integrability in a broad setting.

8.1 Lie Group Actions and the Moment Map

Let G be a Lie group acting in a Hamiltonian fashion on a symplectic manifold (M, ω). By definition, a *Hamiltonian G-action* comes equipped with a *moment map*:

$$\Phi : M \to \mathfrak{g}^*,$$

where $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{g}^*$ its dual. The moment map satisfies the defining property:

$$d\langle \Phi, X \rangle = \omega(\xi_X, \cdot), \quad \forall X \in \mathfrak{g},$$

where ξ_X is the infinitesimal generator of the G-action associated to X.

If $H : M \to \mathbb{R}$ is a G-invariant Hamiltonian (that is, it commutes with the G-action in an appropriate sense), it then Poisson-commutes with all *collective functions* of the form $f \circ \Phi$, for any $f \in C^\infty(\mathfrak{g}^*)$.

8.2 Construction of Integrals Using Group Chains

Thimm's method, introduced by A. Thimm in [17], leverages the moment map to produce a rich supply of integrals in involution. The key idea is to consider a chain of subalgebras

$$\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_k$$

and then restrict the moment map accordingly:

$$\mathfrak{g}^* \to \mathfrak{g}_1^* \to \mathfrak{g}_2^* \to \cdots \to \mathfrak{g}_k^*.$$

At each stage, one considers a set of G_i-invariant functions on $\mathfrak{g}_i^*$, which Poisson-commute. Pulling these back to M via the corresponding reductions of the moment map yields a collection of integrals on M. By carefully choosing the chain of subalgebras and invariants, one can achieve a sufficient number of independent involutive integrals to ensure complete integrability (in the sense of Liouville) of the given Hamiltonian system.

In essence, Thimm's method generalizes the procedure found in classical integrable systems to a much broader class of manifolds, including certain symmetric spaces and Grassmannians.

8.3 Examples and Extensions

Thimm's work sparked considerable interest and led to many new examples of integrable geodesic flows on Riemannian manifolds. For example, Thimm's method can recover the integrability of geodesic flows on:

- Real and complex Grassmannians,
- Distance spheres in complex projective spaces CP^{n+1},
- Homogeneous spaces of the form $SU(n+1)/SO(n+1)$,

In fact, this result (attributed to Thimm) states that these manifolds admit Riemannian metrics with integrable geodesic flows.

9 Duran's Theorem and Z-Manifolds

In the preceding discussions, the integrability of geodesic flows depended, in large part, on the presence of sufficient isometries or symmetry groups. Thimm's method and the construction of integrals in involution typically rely on such structures. However, C. E. Durán's remarkable result extends the realm of integrability to a far broader class of Riemannian manifolds, known as Z-manifolds, without the need for large isometry groups.

Definition 1 (Z-Manifold)**.** *A Riemannian manifold M is called a* Z-manifold *if, for every $p \in M$, all geodesics emanating from p return to p.*

Examples of Z-manifolds include the compact rank-one symmetric spaces (CROSSes):

$$S^2,\ \mathbb{K}P^n \text{ (for } \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}\text{), and the Cayley plane.}$$

The classical examples like spheres and projective spaces fit perfectly into this class and are known to have integrable geodesic flows. Durán showed that this phenomenon is far more general.

Theorem 2 (Durán). *The geodesic flow on any Z-manifold is integrable.*

Durán's theorem is a powerful statement, removing the necessity of having large isometry groups or implementing sophisticated constructions like Thimm's method. Instead, it relies on the fundamental geometric property that all geodesics are *closed* and return to their starting point.

In summary, Durán's theorem shows that Z-manifolds, which include many familiar examples and also more exotic spaces, enjoy integrable geodesic flows. This result significantly broadens the class of manifolds for which Liouville-type integrable structures can be guaranteed.

References

[3] V. Guillemin and S. Sternberg, *On collective complete integrability according to the method of Thimm*, Ergodic Theory and Dynamical Systems, **3**, 1983, 219–230.

[17] A. Thimm, *Integrable geodesic flows on homogeneous spaces*, Ergodic Theory and Dynamical Systems, **1**, 1981, 495–517.

[18] G.P. Paternain and R.J. Spatzier, *New examples of manifolds with completely integrable geodesic flows*, Adv. Math. **108**, 1994, 346–366.

[1] C. E. Durán, *Integrability of geodesic flows on Z-manifolds*, [Reference details if available].