Generalized Co-ordinates for Toric Varieties

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1 Introduction

The contents of this note is mostly taken from chapter 5 of the book [CLS24]. I want to discuss the notions of generalized co-ordinates for toric varieties, which give a different way to present toric varieties, which sometimes give simpler way to describe various geometric notions related to them. Recall, that a toric variety is an algebraic variety X, containing an algebraic torus $\mathbb{T} = \mathbb{G}_m^n$ as a Zariski-dense open subset, such that the action of the torus on itself extends to the whole variety X. This description of a toric variety, although very general, is hard to work with. One usually restricts to normal toric varieties(which we will also do), which can be described by the data of a fan Σ, and have a combinatorial flavor to it. Generalized co-ordinates allow us to describe toric varieties, as the quotient of an action on a certain quasi-affine variety, which is quite natural to describe in terms of the fan.

2 Quotients in Algebraic Geometry

If a variety X, is endowed with an algebraic action of a reductive, algebraic group G(for us the group G will always be a subgroup of an algebraic torus), it is in general a difficult task to define an algebro-geometric object which can play the role of "X/G". To obtain the right answer, let us begin with some definitions of quotients. We will assume the group G is reductive and algebraic throughout.

Definition 2.1. Let G act on a variety X and $\pi : X \to Y$ is a morphism of varieties, constant on G-orbits. We call π a good categorical quotient(gcq) if the following conditions hold:

1. $U \subseteq Y$ open, then the natural map $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$ induces an isomorphism
$$\mathcal{O}_Y(U) \simeq \mathcal{O}_X(\pi^{-1}(U))^G$$
2. $W \subseteq X$ is closed and G-invariant, then $\pi(W) \subseteq Y$ is closed.
3. If $W_1, W_2 \subseteq X$ are closed, disjoint and G-invariant, then $\pi(W_1), \pi(W_2)$ are disjoint in Y.

Remark 2.2.

- If $\pi : X \to Y$ is a gcq, we also denote Y by $X//G$. This has a universal property, that is whenever we have a morphism of varieties, $\phi : X \to Z$ such that $\phi(g \cdot x) = \phi(x)$, $\forall g \in G, x \in X$, there is a unique map $\bar{\phi} : X//G \to Z$ making the following diagram commute :

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & Z \\ & X//G \arrow[ur, dashed, "\exists! \bar{\phi}"'] & \end{tikzcd}$$

If π is a morphism of varieties, such that above factorization exists, then π is called a categorical quotient.

Proposition/Definition 2.3. Let $\pi : X \to X//G$ be a gcq. Then TFAE :

1. All G-orbits are closed in X.
2. Given $x, y \in X$, we have
$$\pi(x) = \pi(y) \iff x, y \text{ lie in the same } G-\text{orbit}$$
3. π induces a bijection
$$\{G - \text{orbits in } X\} \simeq X//G$$
4. The image of the morphism $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is given by the fiber product $X \times_{X//G} X$.

If any, and hence all of the above conditions hold, we say π is a geometric quotient(gq) and we write $\pi : X \to X/G$.

Geometric quotients are the ideal kind of quotients we wish to deal with, but we will see that many quotients that appear naturally, are not geometric, but very close to being geometric. Let's look at two examples. We will use the following fact that if R is a ring with a G-action, and the ring of invariants $R^G = r \in R | g \cdot r = r$ is finitely generated, then $\operatorname{Spec}(R) \to \operatorname{Spec}\left(R^G\right)$ induced by the inclusion $R^G \hookrightarrow R$ is a gcq.

Example 2.4.

1. Let $R = \mathbb{C}[x_1, \cdots, x_n]$, let $\mathbb{C}^\times$ act on it by scalar multiplication
$$\lambda \cdot f(x_1, \cdots, x_n) = f(\lambda x_1, \cdots, \lambda x_n)$$
Then the invariant subring
$$\mathbb{C}[x_1, \cdots, x_n]^{\mathbb{C}^\times} \cong \mathbb{C}$$
as any polynomial fixed by the action must be constant. This happens as the only closed orbit is the orbit of the closed point $0 \in \mathbb{A}^n$. Hence
$$\mathbb{A}^n//\mathbb{C}^\times \cong \operatorname{Spec}(\mathbb{C})$$
but this is very far from a geometric quotient as there are many non-closed orbits.
2. Let $\mathbb{C}^\times$ act on $\mathbb{A}^4 = \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4])$ via
$$\lambda \cdot (a_1, a_2, a_3, a_4) = (\lambda a_1, \lambda a_2, \lambda^{-1} a_3, \lambda^{-1} a_4)$$
In this case, the invariant subring is
$$\mathbb{C}[x_1, \cdots, x_4]^{\mathbb{C}^\times} = \mathbb{C}[x_1x_3, x_2x_4, x_1x_4, x_2x_3]$$
Thus $\operatorname{Spec}\left(\mathbb{C}[x_1, \cdots, x_4]^{\mathbb{C}^\times}\right) = V(xy - zw) \subseteq \mathbb{A}^4$. This is again a gcq. It is not a geometric quotient, as the preimage of $\pi : \mathbb{A}^4 \to V(xy - zw)$ at $(0, 0, 0, 0)$, consists of infinitely many orbits. More directly, $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ have the same image under π but they don't lie in the same $\mathbb{C}^\times$-orbit. But if we consider $U = V(xy - zw) \setminus \{(0, 0, 0, 0)\}$, it is not difficult to see that
$$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$$
is a geometric quotient. This is not a geometric quotient, but is very close to it.

This inspires us to make the following definition :

Proposition/Definition 2.5. Let $\pi : X \to X//G$ be a gcq. Then TFAE :

1. $U \subseteq X$ is a G-invariant Zariski dense open subset, such that $G \cdot x$ is closed in X for every $x \in U$.
2. $X//G$ has a Zariski dense open subset U, such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient.

If any and hence both of the equivalent conditions hold, we say π is an almost geometric quotient(agq).

Remark 2.6. Thus the first quotient in Example 2.4 is not an almost geometric quotient, but the second one is.

3 Toric Varieties as (Almost) Geometric Quotients

Following section 5.1 of [CLS24], in this section we will try to give a description of toric varieties coming from a fan as a geometric quotient

$$X_\Sigma \cong (\mathbb{C}^r \setminus Z)//G$$

for appropriate choices of affine space $\mathbb{C}^r$, a closed subvariety Z inside the affine space, and a reductive group G.

Let us assume that the toric variety X_Σ has no torus factors. It can be checked that that is equivalent to saying the one-dimensional rays u_ρ, $\rho \in \Sigma(1)$ generate the vector space $N_\mathbb{R}$. In such a situation we obtain a short exact sequence of abelian groups -

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \to \mathrm{Cl}(X_\Sigma) \to 0$$

Now if we apply $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^\times)$ to this sequence, this functor is an exact functor, as $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^\times)$ is left exact, and $\mathbb{C}^\times$ is a divisible abelian group, and hence an injective object in the category of abelian groups, making the functor right exact as well. Applying this we get a new exact sequence :

$$1 \to \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^\times) \to (\mathbb{C}^\times)^{\Sigma(1)} \to \mathbb{T}_N \to 1$$

T_N is the torus living inside the toric variety X_Σ. If we denote by $G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^\times)$, then we get a short exact sequence of affine algebraic groups

$$1 \to G \to (\mathbb{C}^\times)^{\Sigma(1)} \to \mathbb{T}_N \to 1$$

Denote $\mathbb{C}^{\Sigma(1)} = \mathrm{Spec}\,(S)$ where $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$, this ring is also called the total co-ordinate ring of X_Σ. The co-ordinates x_ρ are the so-called generalized co-ordinates on X_Σ, but unlike usual co-ordinates, we look at them upto G-action. Now we need to define the exceptional subset Z. For every cone $\sigma \in \Sigma$, denote

$$\widehat{x_\sigma} = \prod_{\rho \notin \sigma(1)} x_\rho$$

where $\sigma(1)$ denotes the rays appearing in the cone σ. Then we define the irrelevant ideal

$$B(\Sigma) := (\widehat{x_\sigma} : \sigma \in \Sigma) \subseteq S, Z(\Sigma) := V(B(\Sigma))$$

Notice to compute $B(\Sigma)$, it is enough to look at the maximal cones, as the other monomials are multiples of the monomials corresponding to maximal cones. Now $(\mathbb{C}^\times)^{\Sigma(1)}$ acts on $\mathbb{C}^{\Sigma(1)}$ by diagonal matrices, and one can induce an action on the complement $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$. Thus $G \subseteq (\mathbb{C}^\times)^{\Sigma(1)}$ also acts. We construct a new fan, $\widehat{\Sigma}$ in $\mathbb{R}^{\Sigma(1)}$ as follows :

- Consider the lattice $\mathbb{Z}^{\Sigma(1)} = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}e_\rho$.
- For every cone $\sigma \in \Sigma$, define

$$\widehat{\sigma} = \mathrm{Cone}(e_\rho : \rho \in \sigma(1)) \subset \mathbb{R}^{\Sigma(1)}$$

- $\widehat{\Sigma} = \{\tau : \tau \preceq \widehat{\sigma} \text{ for some } \sigma \in \Sigma\}$

Now one only needs to notice that $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) = X_{\widehat{\Sigma}}$, the toric variety corresponding to the new fan. We also have a natural map $\mathbb{Z}^{\Sigma(1)} \to N$, sending $e_\rho \mapsto u_\rho$, compatible with $\widehat{\Sigma}, \Sigma$. Thus we obtain a toric morphism $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) = X_{\widehat{\Sigma}} \to X_\Sigma$, which is constant on G-orbits. This is the content of Proposition 5.1.9 of [CLS24]. Now we are ready to state the main theorem :

Theorem 3.1. [CLS24, Theorem 5.1.11] Let X_Σ be a toric variety with no torus factors, and $\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma$ then

1. π is an almost geometric quotient for the action of G on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ and thus

$$X_\Sigma \cong (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))//G$$

2. π is a geometric quotient $\iff$ Σ is simplicial.

Proof. We will give a short sketch of the main ideas of the proof. For more details, we refer to the book. Consider the open U_σ for each cone. Then one can show that $\pi|_{\pi^{-1}(U_\sigma)} : \pi^{-1}(U_\sigma) \to U_\sigma$ is actually a toric morphism $\pi_\sigma : U_{\widehat{\sigma}} \to U_\sigma$. It can be shown that π_σ is gcq, and since the property of being a gcq is local, so is π. The next step is to show that π_σ is a gq if and only if σ is a simplicial cone. Once we have this, to prove the theorem, consider the subfan $\Sigma' \subset \Sigma$ consisting of the simplicial cones. Then $X_{\Sigma'} \subset X_\Sigma$ is open, and furthermore, $\Sigma'(1) = \Sigma(1)$, so both toric varieties have the same total co-ordinate ring S, and reductive group G.

$$\pi^{-1}(X_{\Sigma'}) = \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma') = \bigcup_{\sigma \in \Sigma'} U_{\widehat{\sigma}}$$

Then by what we have already obtained, π_σ is a gq for $\sigma \in \Sigma'$. Hence $\pi|_{\pi^{-1}(X_{\Sigma'})}$ is also a geometric quotient. This proves the theorem. □

Now so far we assumed there are no torus factors. Now let us remove this condition. Suppose, $u_\rho, \rho \in \Sigma(1)$ spans a proper subspace $L_\mathbb{R}$ of $N_\mathbb{R}$. Denote $L = L_\mathbb{R} \cap N$. Choose a complement $N \cong L \oplus P$, let rank(P)= r. Now the same fan Σ gives a fan in $L_\mathbb{R}$. We can write

$$X_\Sigma \cong X_{\Sigma,L} \times (\mathbb{C}^\times)^r$$

where $X_{\Sigma,L}$ is free of torus factors. Also $B(\Sigma, L) = B(\Sigma, N)$ and $Z(\Sigma, L) = Z(\Sigma, N)$. Thus we can apply the previous theorem and write

$$X_{\Sigma,L} \cong (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))//G$$

Following this, one can write

$$X_\Sigma \cong X_{\Sigma,L} \times (\mathbb{C}^\times)^r \cong (\mathbb{C}^{\Sigma(1)+r} \setminus Z'(\Sigma))//G$$

where $Z'(\Sigma) = (Z(\Sigma) \times \mathbb{C}^r) \cup (\mathbb{C}^{\Sigma(1)} \times V(x_1 \cdots x_r))$. Thus we still have a presentation even if the toric variety has torus factors.

Remark 3.2.

1. There are some differences between the case where there are no torus factors and the general case. In the general case, the presentation is non-canonical, as it depends on the choice of a complement P.

2. Also in this case $Z'(\Sigma)$ has codimension 1, but previously $Z(\Sigma)$ always has codimension ≥ 2.

Example 3.3.

1. Consider $\mathbb{P}^n$. The fan Σ has $\Sigma(1) = \{e_0, e_1, \cdots, e_n\}$ where $e_1, \cdots, e_n$ are the standard basis for $\mathbb{R}^n$, and $e_0 + e_1 + \cdots + e_n = 0$. Then $S = \mathbb{C}[x_0, \cdots, x_n]$. There are $n+1$ maximal cones, σ_i, then $B(\Sigma) = (x_0, \cdots, x_n)$ and $Z(\Sigma) = V(B(\Sigma)) = \{0\}$. Since the fan is simplicial, we obtain a geometric quotient
$$\mathbb{P}^n \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$$

2. Consider the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, q_1, \cdots, q_n)$ where q_i's have no common factors. The images of the standard basis e_i in N give elements u_i, and let Σ be the fan generated by proper subsets of $\{u_0, \cdots, u_n\}$. We also have $\sum_{i=0}^n q_i u_i = 0$. Note $\mathbb{C}^{\Sigma(1)} = \mathbb{C}^{n+1}$ and $Z(\Sigma) = \{0\}$ by the previous argument. Note the class group exact sequence takes the form
$$0 \to M \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0$$
where the map $\mathbb{Z}^{n+1} \to \mathbb{Z}$ is given by $(v_0, \cdots, v_n) \mapsto \sum_{i=0}^n q_i v_i$. Thus after taking characters, we get
$$1 \to \mathbb{C}^\times \to (\mathbb{C}^\times)^{n+1} \to T_N \to 1$$
where the first inclusion is $\mathbb{C}^\times \to (\mathbb{C}^\times)^{n+1}$ given by $\lambda \mapsto (\lambda^{q_0}, \cdots, \lambda^{q_n})$. Thus the group $G = \{(\lambda^{q_0}, \cdots, \lambda^{q_n}) : \lambda \in \mathbb{C}^\times\}$. Note the fan is simplicial, so we get again a geometric quotient
$$\mathbb{P}^n(q_0, \cdots, q_n) \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$$
where $\mathbb{C}^\times$ acts by
$$\lambda \cdot (a_0, \cdots, a_n) = (\lambda^{q_0} a_0, \cdots, \lambda^{q_n} a_n)$$
This way we recover the quotient description of the weighted projective space.

4 Quasi Coherent Sheaves on Toric Varieties

Finally, let us apply what we have learnt to understand quasicoherent sheaves on toric varieties. To start off, note that the ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ is graded by the class group $\mathrm{Cl}(X_\Sigma)$. For any monomial $x^a = \prod_{\rho \in \Sigma(1)} x_\rho^{a_\rho}$ we can define

$$\deg(x^a) = \left[\sum_{\rho \in \Sigma(1)} a_\rho D_\rho \right] \in \mathrm{Cl}(X_\Sigma)$$

For $\beta \in \mathrm{Cl}(X_\Sigma)$, we will denote by S_β the β-graded piece. Let $M = \bigoplus_{\beta \in \mathrm{Cl}(X_\Sigma)} M_\beta$ is called a graded S-module if it is an S-module, satisfying $S_\alpha \cdot M_\beta \subseteq M_{\alpha+beta}$. We can associate a quasicoherent sheaf to a graded module.

Proposition 4.1. There exists a quasicoherent sheaf $\widetilde{M}$ on X_Σ such that $\sigma \in \Sigma$, such that for every $\sigma \in \Sigma$,

$$\Gamma(U_\sigma, \widetilde{M}) = (M[\widehat{x_\sigma}^{-1}])_0$$

If M is finitely generated, then $\widetilde{M}$ is coherent.

We also have a shifted module $S(\alpha)$, such that the β-graded piece is given by

$$(S_\alpha)_\beta = S_{\alpha+\beta}$$

It can be shown that

$$\widetilde{S(\alpha)} = \mathcal{O}_{X_\Sigma}(\alpha)$$

which is the line bundle representing the element α in $\mathrm{Cl}(X_\Sigma)$. For a quasicoherent sheaf F, we can thus define $F(\alpha) := F \otimes \widetilde{S(\alpha)}$. Now associate a graded S-module

$$\Gamma_\bullet(F) = \bigoplus_{\beta \in \mathrm{Cl}(X_\Sigma)} \Gamma(X_\Sigma, F(\beta))$$

Then the following theorem holds :

Theorem 4.2. [CLS24, Proposition 6.A.3] Let F be a quasicoherent sheaf on X_Σ. Then $F \cong \widetilde{\Gamma_\bullet(F)}$.

Remark 4.3. The map $M \mapsto \widetilde{M}$ is surjective, as in the above theorem. But it is not injective. For example, if X_Σ is a smooth toric variety, M a finitely generated graded S module, then

$$\widetilde{M} = 0 \iff B(\Sigma)^\ell M = 0 \text{ for } \ell >> 0$$

References

[CLS24] David A Cox, John B Little, and Henry K Schenck. *Toric varieties*. Vol. 124. American Mathematical Society, 2024.