Notes on Lagrangian Pinwheel and Markov Equation

Haosen Wu

December 10, 2024

The goal for today's talk is to give a review of work from Evans-Smith(2) using symplectic orbifold replacement of Weinstein neighborhood of singular Lagrangians "visible" in an almost toric fibration which represents a smooth symplectic manifold and then use orbifold adjunction formula to study curves inside of the orbifold replacement, which in turn gives Markov equation depending on the topology of these immersed singular Lagrangians (called Pinwheel).

1 Topological Object: Lagrangian Pinwheel

Let D denote the unit disc in $\mathbb{C}$ and $\sim_m$ denote the equivalence relation on D which identifies z and z' by $(z/z')^m = 1$ for $z, z' \in \partial D$. The quotient space $D/\sim_m$ is the CW-complex which we call the *pinwheel* P_m. (Draw a neighborhood of arc of ∂D in Pinwheel).

A *Lagrangian* (p,q)*-pinwheel* is essentially topologically given above but some specialization in a symplectic manifold (X,ω), which is a smooth Lagrangian immersion $f \colon D \looparrowright X$ such that, given p, q two positive coprime integers, satisfying:

- $f|_{D \setminus \partial D}$ is an embedding;
- $f(x) = f(y)$ if and only if $x, y \in \partial D$ with $x \sim_p y$;
- if $f(x) = f(y)$ then $f_*(T_x D) \neq f_*(T_y D)$.

notice here the number q not used in definition is essentially the winding number of the disc boundary in some trivialization of the Lagrangian 2-planes in $T_{f(x)}X$-bundle over $D \backslash 0$.

2 Topology of Lagrangian Pinwheel and Connection to Almost Toric Fibration

Here we use series of examples to illustrate topology and its connection to ATF (Almost Toric Fibration):

2.1 Obtain Pinwheels from Smoothing Orbifold: A Local Model

2.1 Obtain Pinwheels from Smoothing Orbifold: A Local Model

Example 2.1.1. Let p, q be coprime integers with $1 \leq q < p$. Let $\Gamma_{p,q}$ be the action of μ_{p^2} with weights $(1, pq-1)$. The surface $\mathbb{C}^2/\Gamma_{p,q}$ has a singularity of type $\frac{1}{p^2}(1, pq-1)$ at the origin. We can embed the surface $\mathbb{C}^2/\Gamma_{p,q}$ into the 3-fold $\mathbb{C}^3/\mu_p$ of type $\frac{1}{p}(1, -1, q)$, as the subvariety $\{xy = z^p\}$. There exists a $\mathbb{Q}$-Gorenstein smoothing

$$\mathcal{S}_{p,q} = \{xy = z^p + t\} \subset \mathbb{C}^3/\mu_p \times \mathbb{C}_t$$

of $\mathbb{C}^2/\Gamma_{p,q}$. Pick $t \neq 0$ and let $S_{p,q}$ be the fibre of this $\mathbb{Q}$-Gorenstein smoothing over t. This is the Milnor fibre of $\mathbb{C}^2/\Gamma_{p,q}$. We can show tahat $S_{p,q}$ is a rational homology ball.

Use a particular presentation of $S_{p,q}$ as a quotient of a Lefschetz fibration we can see that variety $S_{p,q}$ contains a Lagrangian (p, q)-pinwheel $L_{p,q}$. The variety $S_{p,q}$ admits a Liouville form for which it is the symplectic completion of a compact Stein domain $B_{p,q}$ and thus $B_{p,q}$ has $L_{p,q}$ as its Lagrangian skeleton.

2.2 Globalization of Local Model: Hacking-Prokhorov's $\mathbb{Q}$-Gorenstein smoothing of any Fano surface with log canonical singularities, and connection to *mutation*

We start with the orbifold weighted projective space, show that the smoothing of this object results in $\mathbb{C}P^2$ with unique ATF associated with the procedure. During smoothing on each orbifold point we can view the procedure as what described in the local model above, thus this explain the name "globalization".

Example 2.2.1. Hacking and Prokhorov(3) show that if (p_1, p_2, p_3) is a Markov triple then the smoothing of $\mathbb{C}P(p_1^2, p_2^2, p_3^2)$ is a Fano surface with $K^2 = 9$, hence biholomorphic to $\mathbb{C}P^2$. Suppose that (p_1, p_2, p_3) is a Markov triple and let $p_3' = 3p_1p_2 - p_3$. The triple (p_1, p_2, p_3') is again a Markov triple. We call the transition $(p_1, p_2, p_3) \to (p_1, p_2, p_3')$ a *mutation*. The $\mathbb{Q}$-Gorenstein deformation

$$\{z_0z_1 - (1-t)z_2^{p_3'} - tz_3^{p_3} = 0\} \subset \mathbb{C}P(p_1^2, p_2^2, p_3, p_3')$$

connects $\mathbb{C}P(p_1^2, p_2^2, p_3^2)$ at $t = 0$ to $\mathbb{C}P(p_1^2, p_2^2, (p_3')^2)$ at $t = 1$, both have of $K^2 = 9$. Since any Markov triple (p_1, p_2, p_3) can be related to $(1, 1, 1)$ by a sequence of mutations, where $\mathbb{C}P(1, 1, 1) = \mathbb{C}P^2$ with $K^2 = 9$, the $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{C}P(p_1^2, p_2^2, p_3^2)$ thus is $\mathbb{C}P^2$ for any Markov triple (p_1, p_2, p_3).

The weighted projective space $\mathbb{C}P(p_1^2, p_2^2, p_3^2)$ is therefore a toric orbifold with at most three orbifold singularities

$$\frac{1}{p_i^2}(1, p_iq_i - 1), \qquad i = 1, 2, 3.$$

For each Markov number p, we obtain a Lagrangian pinwheel $L_{p,q} \subset \mathbb{C}P^2$ which is the vanishing cycle of the $\frac{1}{p^2}(1, pq-1)$ singularity.

The singularity $\mathbb{C}^2/\Gamma_{p,q}$ is toric and its moment polytope can be modified by a nodal trade at the singularity to give an almost toric fibration on the smoothing $B_{p,q}$. The vanishing thimble for the singular fibre gives the Lagrangian Pinwheel. Putting the modification together for all orbifold point, we obtain an affine base diagram for the ATF of this smoothed weighted projective space $\mathbb{C}P(p_1^2, p_2^2, p_3^2)$ (Draw an ATF).

2.3 Topology of $B_{p,q}$ and its skeleton *Pinwheel* $L_{p,q}$, by some explicit calculation on the Pinwheel

We give some topological property of $B_{p,q}$ along with the notation: Suppose that (X, ω) is a symplectic 4-manifold with $H_1(X;\mathbb{Z}) = 0$, $H_2(X;\mathbb{Z}) = \mathbb{Z}$, and $B_i \subset X$, $i = 1, \dots, N$, is a collection of pairwise disjointly embedded symplectic rational homology balls $B_i \cong B_{p_i,q_i}$. Let Σ_i denote ∂B_i, let $B = \coprod_{i=1}^N B_i$, $\Sigma = \coprod_{i=1}^N \Sigma_i$ and $V = X \setminus B$. The rational ball $B_{p,q}$ is homotopy equivalent to the pinwheel $L_{p,q}$ which is a Moore space $M(\mathbb{Z}/(p), 1)$. We have $H_1(\Sigma_{p,q};\mathbb{Z}) = \mathbb{Z}/(p^2)$ and $H_1(B_{p,q};\mathbb{Z}) \cong \mathbb{Z}/(p)$. Then we have the following:

Lemma 2.3.1. *The first Chern class $c_1(B_{p,q}) \in H^2(B_{p,q};\mathbb{Z}) \cong \mathbb{Z}/(p)$ is primitive.*

Lemma 2.3.2. *Let $\jmath\colon V \to X$ be the inclusion map, $\Delta = \prod_{i=1}^N p_i$ and L_i denote the Lagrangian pinwheel in B_i. We define $\mathcal{E} \in H_2(V, \Sigma;\mathbb{Q})$ to be the unique element such that $m\mathcal{E}$ generates the lattice $H_2(V, \Sigma;\mathbb{Z})/T \subset H_2(V, \Sigma;\mathbb{Q})$, where $T = \mathbb{Z}/(\Delta/m)$. The map*

$$\jmath^*\colon H^2(X;\mathbb{Q}) \to H^2(V;\mathbb{Q}),$$

by Poincaré duality sends a generator $H \in H^2(X;\mathbb{Z})$ to $\Delta\mathcal{E} \in H_2(V, \Sigma;\mathbb{Q})$.

3 Orbifold Adjunction and SFT analysis of low degree curves

This part is essentially the proof of the relation of Markov triples and Lagrangian Pinwheels, and explain the *mutations* coming from different ATF of $\mathbb{C}P^2$.

Using local models, we now replace a symplectic rational homology ball $B_{p,q} \subset X$ by an orbifold singularity to obtain relating symplectic orbifold, as follows:

$$\hat{X} = V \cup_{\phi_i(\Sigma_i) \cong \hat{\Sigma}_i} \coprod_{i=1}^N \hat{B}_i.$$

Here $B_i \subset X$, $i = 1, \dots, N$ is a collection of pairwise disjoint symplectic embeddings of rational homology balls $B_i \cong B_{p_i,q_i}$. Σ_i denotes the boundary ∂B_i and $\Sigma = \bigcup_{i=1}^N \Sigma_i$. $V = X \setminus \coprod_{i=1}^N B_i$. $\hat{\Sigma}_i = S^3/\Gamma_{p_i,q_i} \subset \mathbb{C}^2/\Gamma_{p_i,q_i}$. $\hat{B}_i$ denotes the compact component of $\left(\mathbb{C}^2/\Gamma_{p_i,q_i}\right) \setminus \hat{\Sigma}_i$. (This is the key observation and thus bridges the analysis of pinwheel by the adjunction formulae later so that we can derive the geometry of pinwheels and Markov triples).

Modulo some details in constructing corresponding orbifold symplectic form and orbifold holomorphic curves, we can proceed to use Weimin Chen's orbifold adjunction formula(1) on the low degree curves (represented by parametrized holomorphic curves) living in the symplectic orbifold:

Theorem 3.0.1. *Let* $f\colon \hat{S} \to \hat{X}$ *be a somewhere-injective orbifold holomorphic curve representing a homology class* $C \in H_2(\hat{X};\mathbb{Q})$. *Let* $g_{|\hat{S}|}$ *denote the genus of the underlying smooth Riemann surface. Then*

$$1 + \frac{C \cdot C - c_1(\hat{X}) \cdot C}{2} = g_{|\hat{S}|} + \frac{1}{2}\sum_{z \in Z}\left(1 - \frac{1}{|H_z|}\right) + \sum_{z \in \hat{S}} k_z + \sum_{\substack{\hat{S} \ni z \neq z' \in \hat{S} \\ f(z) = f(z')}} k_{[z,z']}. \tag{3.1}$$

where $\hat{S}$ *is some orbifold Riemann surface,* H_z *the isotropy group of* $\hat{S}$ *at* $z \in \hat{S}$ *and* G_z *be the isotropy group of* $\hat{X}$ *at* $f(z)$, *finally* $k_z k,_{[z,z']}$ *is defined as some sum of the intersection number of local lifting of these somewhere-injective orbifold holomorphic curve in the local covering of the orbifold.*

Using the topological fact in Section 2.3 we can simplify the adjunction formula, especially on the term involved intersection numbers on the left and counting terms on the first two terms of right hand side. We can show that by simply apply this variation of the adjunction formula the only cases we have to consider are when $|Z| = 1, 2$ for the low degree curves (Z is the set of orbifold point in the parametrizing Riemann surfaces for these low degree curves).

Depending on the cardinality $|Z|$, using neck-stretching from SFT analysis for curves in orbifold. The adjunction formula above for these $|Z| = 1, 2$ curves will degenerate to the Markov equation, where the numbers b and c have the following geometric interpretation: in a local lift of the curve to $\mathbb{C}^2$ in a neighbourhood of the orbifold point $\mathbb{C}^2/\Gamma$, the link of the orbifold point of the curve is a (b^2, c^2)-torus knot.

References

[1] Weimin Chen, *Orbifold adjunction formula and symplectic cobordisms between lens spaces*, Geometry amp; Topology **8** (2004), no. 2, 701–734.

[2] Jonathan Evans and Ivan Smith, *Markov numbers and lagrangian cell complexes in the complex projective plane*, Geometry amp; Topology **22** (2018), no. 2, 1143–1180.

[3] Paul Hacking and Yuri Prokhorov, *Smoothable del pezzo surfaces with quotient singularities*, Compositio Mathematica **146** (2009), no. 1, 169–192.