Tropical Lagrangian

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1 Introduction

Let $p : M_\Delta \to \Delta$ be the moment map of the symplectic toric manifold associated to the Delzant polytope Δ. We can consider p as a Lagrangian torus fibration, which is a special case of almost toric fibrations, ATF. The main goal of this exposition is to give a general way to consturct Lagrangian submanifolds inside the total space of ATF, called tropical Lagrangians, following Mikhalkin[1]. We will first review some basics of tropical curves and visible Lagrangian submanifolds. Then the specific constructions of the latter will give rise to a dictionary between tropical curves in the base diagram of an almost toric fibration and tropical Lagrangian submanifolds of the corresponding symplectic manifold.

2 Tropical Curves and Lagrangian Realizability

We give the minimal amount of related definitions in tropical geometry for completeness of our construction in section 3. For ones who wish to avoid suffering reading definitions, we can just consider tropical curves as $3-$valent connected nontrivial graphs inside a polyhedral domain, which will later be associated with a Lagrangian submanifold via explicit geometric description. Take an $n-$dimensional lattice Λ. Let $A = \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.1: If $\overline{\Gamma}$ is a topological space homeomorphic to a finite graph, and $\partial\Gamma \subset \overline{\Gamma}$ is its $1-$valent vertices while V_Γ is the set of vertices of valence greater than or equal to 2. Define $\Gamma = \overline{\Gamma} \backslash \partial\Gamma$. A **tropical curve** is a topological space Γ as defined above equipped with an inner complete metric.

Definition 2.2: An immersion $h : \Gamma \to A$ is called **tropical** if:
(a) For any edge $e \subset \Gamma$, a point $x \in e$ and a unit tangent vector $u \in T_x e$ the restriction $h|_e$ is a smooth map such that $dh_x(u) \in \Lambda \subset T_{h(x)}A$. We denote $dh(e) = dh_x(u)$.
(b) (balancing condition)For any vertex $v \in V_\Gamma$, $\Sigma_{v \in \bar{e}} dh(e)$ where the orientation of e are chosen to be away from v.
A tropical immersion is locally flat if, in addition, for a collection of edges e_j adjacent to v has a $2-$dimensional linear span by $dh(e_j)$ inside $\Lambda \otimes \mathbb{R}$. Moreover,

the intersection $h(\Gamma) \cap \Delta \subset \Delta \subset A$ is called a **locally flat tropical curve in a polyhedral domain** Δ.

Definition 2.3: A tropical immersion $h : \Gamma \to A$ is called **primitive** if
(a) The tropical curve Γ is connected, $3-$valent with nonempty vertices set.
(b) For any edge e, $dh(e)$ is a primitive element of the lattice Λ.
(c) $x \neq y \in \Gamma$ with $h(x) = h(y) \implies x, y \in \Gamma \backslash V_\Gamma$.
We call the image $h(\Gamma)$ a primitive tropical curve in A.

Definition 2.4: Considering Γ_Δ a topological space homeomorphic to a connected graph, an immersion $h_\Delta : \Gamma_\Delta \to \Delta \subset A$ is called $\Delta-$**tropical** if there is a tropical immersion $h : \Gamma \to A$ such that $\Gamma_\Delta \subset \Gamma$ and $h|\Gamma_\Delta = h_\Delta$ with $h^{-1}(\partial\Delta)$ is a finite set disjoint from V_Γ. A $\Delta-$tropical immersion h_Δ is **primitive** if h can be chosen primitive and $h^{-1}(x)$ is a singleton for any $x \in \partial\Delta$. A subset $C \subset \Delta$ is a primitive tropical curve in Δ if there exists a primitive $\Delta-$tropical immersion with $C = h_\Delta(\Gamma_\Delta) \cap \Delta$.

Definition 2.5: The **boundary points** a of primitive tropical curve $C \subset \Delta$ are the points in $\partial C = C \cap \partial\Delta$. Note that each boundary point belongs to a unique $(n-k)-$dimensional face of the polyhedral domain Δ. We call k the **codimension** of the boundary point. When $k = 1$ for $x \in \partial C$, e_x is the edge containing $h^{-1}(x)$, and $\delta_x \subset \partial\Delta$ the face containing x. We define the **boundary momentum** $p(x)$ to be the **tropical intersection number** between $h(e_x)$ and Δ_x in A, which is the index of the sublattice generated by $dh(e_x)$ and the elements of Λ parallel to Δ_x in Λ.

Definition 2.6: A boundary point $x \in \Delta_1 \cap \Delta_2$ of codimension 2 is called a **bissectrice** if the boundary momenta $\delta_i = 1$ for $i = 1, 2$ with $\Delta_1 \cap \Delta_2$ giving the codimension 2 facet where x lives in. δ_i's are defined to be the boundary momentum $p(x)$ as x considered a point in Δ_i.

Definition 2.7: A primitive tropical curve in Δ is called **even** if
(a) all of its boundary points are of codimension at most 2.
(b) all of its codimension 1 boundary points have boundary momenta equal to 2.
(c) all of its codimension 2 boudnary points are bissectrice points.

Definition 2.8: The **multiplicity** of v is the area of the parallelogram spanned by vectors $dh(e_1)$ and $dh(e_2)$

$$m(v) = |dh(e_1) \wedge dh(e_2)|$$

The self-intersection number of v is defined by $\delta(v) = \frac{m(v)-1}{2}$

Definition 2.9: We say that $C \subset \Delta$ is Lagrangian realizable if there exists a family of proper Lagrangian immersions $v_\epsilon : L \to M_\Delta$ smoothly depending on

an arbitrary small parameter $\epsilon > 0$ with
(a) we have $\mu_\Delta(v_\epsilon(L)) \subset C \cup U_\epsilon(V_C)$, where V_C are vertices of C and U_ϵ denotes an $\epsilon-$disk in the 2$-$dimensional affine subspace containing the 3 edges containing x, which is valid by balancing condition. Furthermore, for any $x \in C \backslash (U_\epsilon(V_C) \cup \partial C \cup \Sigma(C)$, $L \cap \mu_\Delta^{-1}(x)$ is an affine subtorus of $\mu_\Delta^{-1}(x)$, where $\Sigma(C)$ is the self intersection set of C which can be proven to be finite.
(b) For every vertex $v \in V_\Gamma$, $(\mu_\Delta \circ v_\epsilon)^{-1}(U_\epsilon(x))$ is homeomorphic to the product of a pair of pants with a subtorus T^{n-2}. In particular, we have a diffeomorphism of pairs in a neighbourhood of v,

$$(\mu_\Delta^{-1}(U_\epsilon(v)), (\mu_\Delta \circ v_\epsilon)^{-1}(U_\epsilon(x))) \cong ((\mathbb{C}^*)^2, \phi_v(P_\delta(v))) \times T^{n-2}$$

where $\phi_v : P_\delta(v)) \to (\mathbb{C}^*)^2$ is an embbedding whose image is an irreducible immersed rational holomorphic curve with 3 punctures and $\delta(v)$ ordinary nodes of positive self-intersections, i.e. a pair of pants with $\delta(v)$ many nodes.

Theorem: Any even primitive tropical curve C in a Delzant polyhedral domain Δ is Lagrangian realizable.

Instead of giving the proof of the theorem in full generality, we gives concrete construction in dimension 4 and survey some examples in the remaining of the notes.

3 Visible Lagrangians

We are going through some basic ideas of visible lagrangian submanifold in an almost toric fibration.

Theorem-Definition: Consider the integrable Hamiltonian system $H : \mathbb{R}^n \times T^n \to \mathbb{R}^n$, $H(\mathbf{p}, \mathbf{q}) = \mathbf{p}$ where q_i coordinates are modulo 2π and the symplectic form is $\Sigma dp_i \wedge dq_i$. Let L be a Lagrangian submanifold and $H|_L : L \to \mathbb{R}^n$ factor as $H|_L = f \circ g$ with $g : L \to K$ a bundle over a $k-$dimensional manifold K and $f : K \to \mathbb{R}^n$ an embedding. Then K is an affine linear subspace of R^n which is rational with respect to the lattice $2\pi\mathbb{Z}^n$. Such Lagrangian submanifolds are called **visible**.

3.1 visible Lagrangian cylinder

Now we take $n = 2$ and K the p_1-axis. Then $L \cap H^{-1}((p_1, 0)))$ needs to be a circle $\{(q_1, \theta) | \theta \in [0, 2\pi]\}$ for some fixed value q_1. One can take $L = \{(p_1, 0, 0, q_2) \in \mathbb{R}^2 \times T^2\}$. Such cylinder can be constructed for any line in the action domain.

3.2 Schoen-Wolfson cone

Now we restrict to the case when $\mu : X \to \mathbb{R}^2$ is the moment map of a 4$-$dimensional symplectic toric manifold. We want to know how the visible

Lagrangian look like when the 1−dimensional affine submanifold $K \subset \mathbb{R}^2$ hits a vertex of the image of the moment map.

Fix a pair of coprime positive integer (m, n) and consider the ray defined by $K = \{(mt, nt)|t \geq 0\} \subset \mathbb{R}^2$. The corresponding visible Lagrangian is called a Schoen-Wolfson cone $L_{(m,n)}$ parameterized by

$$(s,t) \mapsto \frac{1}{\sqrt{m+n}}(t\sqrt{m}e^{is\sqrt{n/m}}, it\sqrt{n}e^{-is\sqrt{m/n}})$$

for $x \in [0, 2\pi\sqrt{mn}]$ and $t \geq 0$. Note that the image is a topological disk but it is in general singular at the origin except for the case $m = n = 1$. Here we identify the total space $\mathbb{R}^4$ as $\mathbb{C}^2$. This construction gives a visible Lagrangian submanifold over the ray K away from the origin point.

3.3 (n,m)-pinwheel core

Next, we examine what L would look like when the projection sends L to a ray K starts from a point on the edge of the moment polytope. The local model of X is $\mathbb{R} \times S^1 \times \mathbb{C}$, given by coordinates $(p, q, z = x + iy)$ with symplectic form $dp \wedge dq + dx \wedge dy$. The image of the moment map $\mu(p, q, z) = (p, \frac{1}{2}|z|^2)$ is the upper half plane. We consider the ray defined $\{(ms, ns)|s \geq 0\}$. We take the Lagrangian immersion of the half cylinder $(s, t) \in [0, \infty) \times S^1 \mapsto (ms, -nt, \sqrt{2ns}e^{imt})$ whose image under the moment map is the ray K. Note that this immersion is an embedding away from $s = 0$ and is of degree n along the circle $s = 0$. We call the image of the immersion a Lagrangian (n, m)−pinwheel core.

3.4 Vanishing Thimble

When there's a base node in the interior, considered locally as a focus-focus critical value, at which an edge points in the eigendirection for the affine monodromy of the base node. We invoke lemma 6.15 in Evans[2]:

Lemma: Let $H : X \to \mathbb{R}^2$ be an integrable Hamiltonian system with a focus-focus critical point at x. Let B be the set of regular values and $\overline{B}$ be its unversal cover with $I : \overline{B} \to \mathbb{R}^2$ the developing map for the integral affine structure on B coming from action coordinates. Let $b \in \mathbb{R}^2$ be the base node associated to the focus-focus singularity x. Suppose l is a straight ray in $\mathbb{R}^2$ emanating from b in an eigendirection for the affine monodromy around the critical value. Then there's a visible Lagrangian disk living over l.

This Lagrangian disk is called, similar to the situation in Picard-Lefshetz theory, the vanishing thimble for the focus-focus singularity.

4 Tropical Lagrangians in dimension 4

Before we summarize how to realize a tropical curve as a Lagrangian submanifold, one more key ingredient is needed-"the Lagrangian over a $3-$valent vertex".

Lemma: Consider the hyperKähler twist $\tau : \mathbb{R}^2 \times T^2 \to \mathbb{R}^2 \times T^2$ given by $(p_1, p_2, q_1, q_2) \mapsto (p_1, p_2, -q_2, q_1)$. If $C \subset \mathbb{R}^2 \times T^2$ is a complex curve with respect to the complex structures induced by $z_k = x_k + iy_k$, then the image of $\tau(C)$ is a Lagrangian submanifold with respect to $\Sigma dp_i \wedge d1_i$.

Now consider the complex curve $C = \{z_2 = 1 + z_1\} \subset \mathbb{C}^* \times \mathbb{C}^* \cong R^2 \times T^2$ which is diffeomorphic to a pair of pants, a 3-punctured sphere. We take the parametrization $z \mapsto (z, 1+z)$ defined on $\mathbb{C}\backslash\{0, -1\}$. Then coordinate-wisely the Lagrangian L is given by $(p_1, p_2, q_1, q_2) = (ln|z|, ln|1 + z|, -arg(1 + z), arg(z))$.

As $|z| \to 0$, we have $p_2, q_1 \to 0$, thus the Lagrangian L is modeled by the visble Lagrangian cylinder $L_1 = \{(p_1, 0, 0, q_2)|p_1 < 0 q_2 \in [o, 2\pi]\}$ as in section 3.1 near $z = 0$. Similar constructions as $z \to -1, \infty$ gives two other visible Lagrangian cylinders

$$L_2 := \{(0, p_2, q_1, 0)|p_2 > 0, q_1 \in [0, 2\pi]\}$$

$$L_3 := \{(p, p, p, -q)|p > 0, q \in [0, 2\pi]\}$$

Next, we want to modify the Lagrangian submanifold L to L' such that we have $K_1 \cap L' = K_1 \cap L$ and $U_2 \cap L' = U_2 \cap (L_1 \cup L_2 \cup L3)$ for $0 < k_1 < k_2$, $K_i = \{p_1^2 + p_2^2 \leq k_i\}$ and $U_i = (\mathbb{C}^*)^2 \backslash K_i$.

The modification near L_1 starts with identifying a neighbourhood of L_1 as T^*L_1 via Weinstein neighbourhood theorem. In particular, L_1 has coordinates $p_1 = lnr$ and $q_2 = \theta$ and the dual momenta, i.e. the cotangent coordinates, $-q_1$ and p_2. From Lagrangian condition we can extract

$$q_1 = -arctan(\frac{e^{p_1} sin(q_2)}{1 + e^{p_1} cos(q_2)})$$

$$p_2 = \frac{1}{2} ln(1 + 2e^{p_1} cos(q_2) + e^{2p_1})$$

i.e. L_1 is the graph of the $1-$form

$$\beta = \frac{1}{2} ln(1 + 2e^{p_1} cos(q_2) + e^{2p_1} dp_1 + arctan(\frac{e^{p_1} sin(q_2)}{1 + e^{p_1} cos(q_2)}) dq_2$$

In fact, β is both closed and exact. Thus, we can find a primitive $\phi(p_1, q_2)$ with $\beta = d\phi$. Take $\epsilon > 0$, we can pick a cut-off function $\rho(p_1)$ such that

$$\rho = \begin{cases} 1 \; p_1 \leq -ln2 - \epsilon \\ 0 \; p_1 \geq -ln2 + \epsilon \end{cases} \tag{1}$$

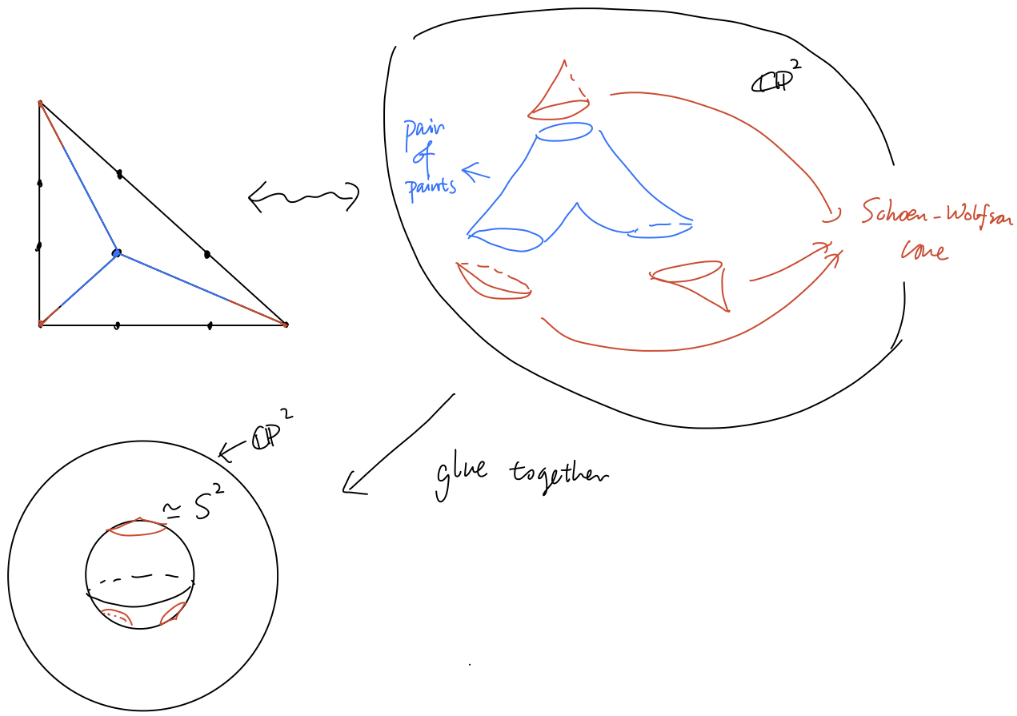
Here we take $k_1 = (ln2 - \epsilon)^2$ and $k_2 = (ln2 + \epsilon)^2$, then the graph Lagrangian $d(\rho\phi)$ is the desired modification. We modify near L_2 and L_3 similarly. The resulted Lagrangian pair of pants have cylinder ends L_i's outside a compact subset. Note that we can make the compact subset arbitrarily small by flowing backward along the radial Liouville vector field of $\mathbb{R}^2$ in the base. This construction gives us:

Theorem(Mikhalkin): Let Γ be a tropical curve in $\mathbb{R}^2$ with only one vertex v and 3 edges e_1, e_2, e_3. For any $\epsilon > 0$, there's a embedded Lagrangian submanifold $L \subset \mathbb{R}^2 \times T^2$ diffeomorphic to the pair of pants such that $U \cap L = U \cap (L_1 \cup L_2 \cup L_3)$ where U is the complement of the ϵ-disk centered at v.

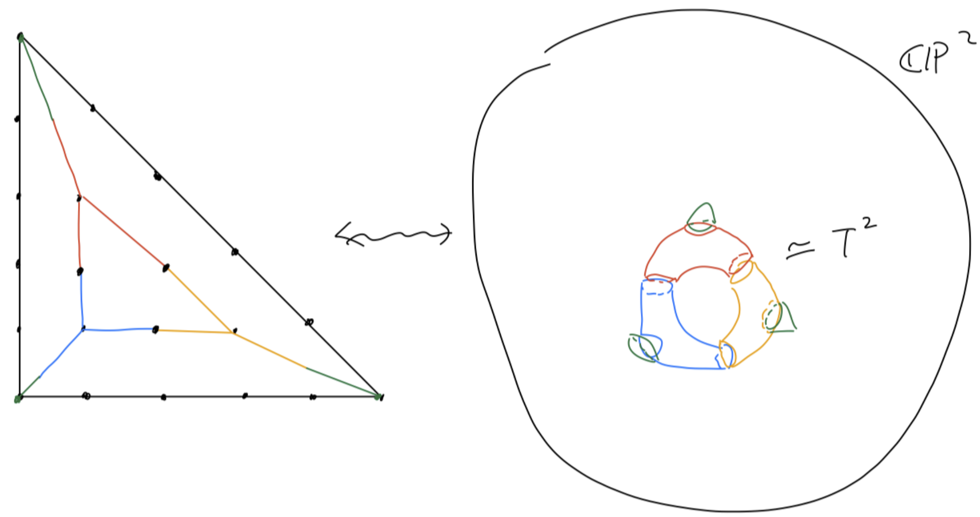
With this in hand, we can summarize to give an alphabet of tropical Lagrangian as follows:

Tropical Curve	Tropical Lagrangian
A trivalent vertex v	An immersed Lagrangian pair of pants with $\delta(v)$ self-intersection
An edge	A Lagrangian cylinder
A ray terminates at a vertex of an ATF base diagram	A Schoen-Wolfson cone
A ray terminates at a codimension 1 boundary point of an ATF base diagram	A (n,m)-pinwheel core
An edge termates at a base node along the eigendirection of the affine monodromy	A Lagrangian disk

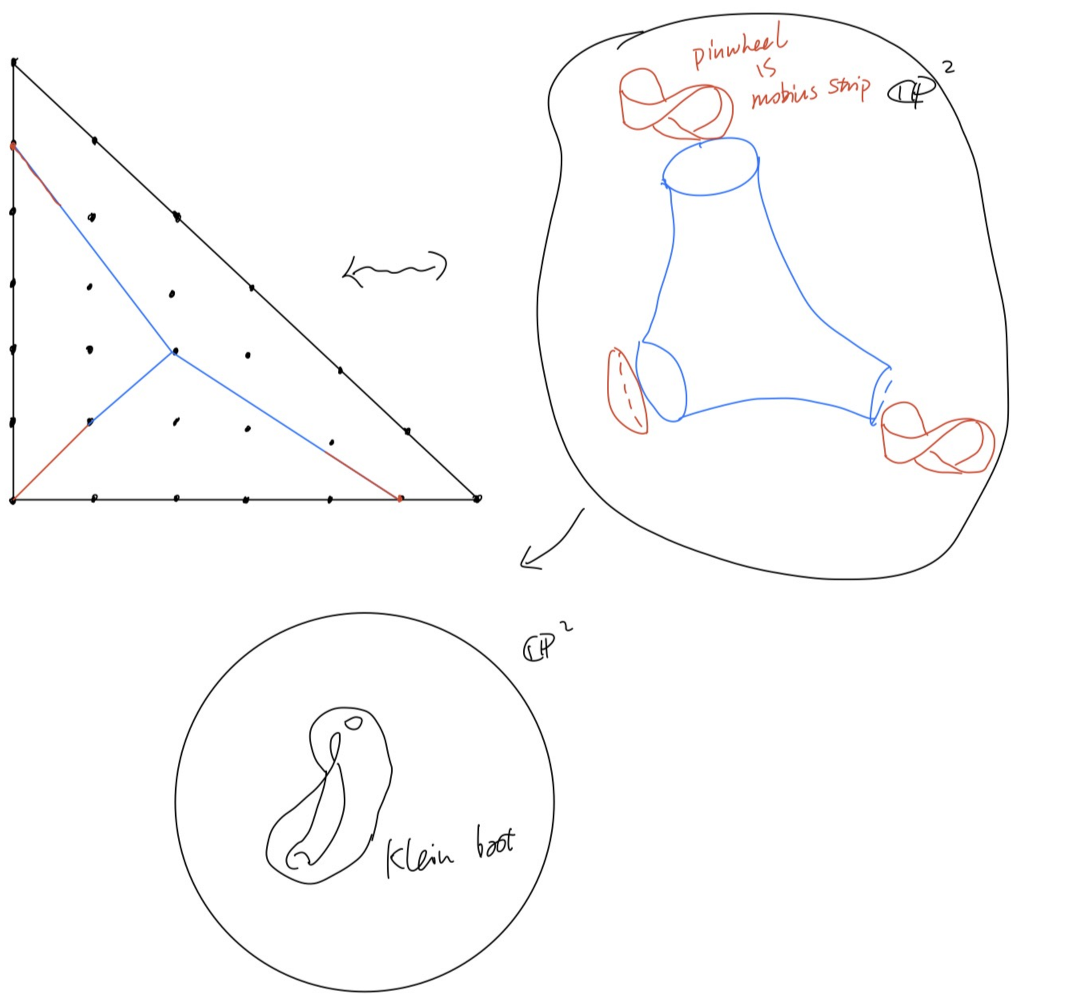
Example 4.1 (Lagrangian sphere): The picture below gives a tropical curve with one vertex and 3 edges ending at the vertices of the moment polytope of $\mathbb{C}P^2$. We apply the above alphabet to construct the associated tropical Lagrangian which is a topological 2−sphere with 2 singularities corresponding 2 the vertices $(3, 0)$ and $(0, 3)$.



Example 4.2 (Lagrangian torus): This example gives a tropical curve with 3 vertices where each vertex has 2 edges connecting with the other 2 vertices and the remaining edge goes to the vertex of the moment polytope of $\mathbb{C}P^2$. Thus, each vertex gives a pair of pants and each edge reaching the vertex of the polytope gives an Schoen-Wolfson cone. Again, only the left bottom cone is nonsingular. Glued together, the tropical Lagrangian is a torus.



Example 4.3 (Lagrangian Klein bottle): The tropical curve in this case has one vertex and 1 edge going to the vertex of the moment polytope and 2 edges ending at the codimension 1 boundary point. The former edge corresponds to a disk and the latter 2 edges are pinwheels diffemorphic to mobius strip. Gluing along the boundary circles of the mobius strips to two ends of the pair of pants and capping by the disk, we obtain a Lagrangian Klein bottle.



References

[1]Mikhalkin, G. (2019) *Examples of tropical-to-Lagrangian correspondence. European Journal of Mathematics, 5(3), 1033–1066. https://doi.org/10.1007/s40879-019-00319-6.*

[2]Evans *Lectures on Lagrangian Torus fibrations. Cambridge University Press..*