GIT : $k = \bar{k}$

A is a finitely generated k-algebra.

Affine case : Let G reductive, $G \curvearrowright \operatorname{Spec} A = X$. Then A^G is also finitely generated

$A^G \xrightarrow{i} A \rightsquigarrow \operatorname{Spec} A \twoheadrightarrow \operatorname{Spec} A^G$. $X /\!/ G := \operatorname{Spec} A^G$ which is a categorical quotient.

In general this is not an orbit space, we will work on an open set of stable points on which π restricts to a geometric quotient.

Quasi-projective case : Let $X \hookrightarrow \mathbb{P}^n_k$ be a projective scheme, its homogenous coordinate ring is a graded ring $R = \bigoplus_{j \geq 0} R_j$ where $R_0 = k$, R is finitely generated as a k-algebra and R_1 generates R.

R depends on X & the embedding $i : X \hookrightarrow \mathbb{P}^n_k$.

There exist such a correspondence : $\{(X, L) \mid X$ polarized projective scheme, L very ample line bundle$\} \xrightarrow[\text{bij}]{\simeq} \{R \mid$ finitely generated graded k-algebra generated in degree 1$\}$

$(X, L) \longmapsto R(X, L) := \bigoplus_{j \geq 0} H^0(X, L^{\otimes j})$

$(\operatorname{Proj}(R), \mathcal{O}(1)) \longleftarrow R$

Problems here : we need to lift the action $G \curvearrowright X$ to an action $G \curvearrowright R(X, L)$ for some line bundle L.

We solve this problem via linearization : Let $G \curvearrowright X$ where X is quasi projective scheme. A linearization of $G \curvearrowright X$ is a line bundle L, $L \xrightarrow{p} X$ projection is G-equivariant and morphism of fibers $L_x \to L_{g \cdot x}$ is linear.

Semistable points : $x \in X$ is called semistable (with respect to L) if $\exists\, j > 0$ and an invariant section $\sigma \in R(X, L^{\otimes j})^G$ such that $\sigma(x) \neq 0$. $X^{ss}(L) = X \setminus V(R(X, L)^G)$ is an open subscheme.

- Let G be a reductive group acting on a projective scheme X with respect to an ample linearization; then GIT quotient $\pi : X^{ss}(L) \longrightarrow X /\!/_L G := \operatorname{Proj} R(X, L)^G$ is a good quotient (categorical, geometric) of the G-action on $X^{ss}(L)$.

Example : Let $G = \mathbb{G}_m \curvearrowright X = \mathbb{P}^1_{\mathbb{C}}$, $t \cdot [x_0 : \cdots : x_1] = [t^{-1} x_0 : t x_1 : \cdots : t x_n]$.

Then we have $R(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^n}(1))^G = \bigoplus_{j \geq 0} \mathbb{C}[x_0, \ldots, x_n]_j^{\mathbb{G}_m} = k[x_0 x_1, \ldots, x_0 x_n]$

$$(\mathbb{P}^n_{\mathbb{C}})^{ss} = \mathbb{P}^n_{\mathbb{C}} \setminus V(x_0 x_1, \ldots, x_0 x_n) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n_{\mathbb{C}} \mid x_0 \neq 0 \ \& \ (x_1, \ldots, x_n) \neq 0\} \simeq \mathbb{A}^n \setminus \{0\}$$

and GIT quotient is $\varphi : (\mathbb{P}^n_{\mathbb{C}})^{ss} \simeq \mathbb{A}^n \setminus \{0\} \longrightarrow X /\!/ G = \operatorname{Proj} k[x_0 x_1, \ldots, x_0 x_n] \simeq \mathbb{P}^{n-1}$

Symplectic quotient:

K Lie group, $K \curvearrowright (X, \omega)$ $(K \xrightarrow{\Psi} Sp(X, \omega))$

A moment map for $K \curvearrowright (X, \omega)$ is a smooth map $\mu : X \to \mathfrak{k}^*$ where $\mathfrak{k} = \mathrm{Lie}(K)$ such that

• 1 $\forall v \in \mathfrak{k}^*$, $\mu^v : X \to \mathbb{R}$, $x \mapsto \langle \mu(x), v \rangle$, v^* is the vector field on X generated by $\{\exp(tv) \mid t \in \mathbb{R}\} \leq K$

$d(\mu^v) = \iota_{v^*}(\omega)$. ($\mu$ lifts the infinitesimal action)

• 2 μ is K-equivariant; $\mu \circ \Psi_a = \mathrm{ad}_a^* \circ \mu$, $\forall a \in K$.

For a coadjoint fixed point $a \in \mathfrak{k}^*$, $X /\!/^{\mathrm{red}}_a K := \mu^{-1}(a)/K$. (Equivariance of μ implies $\mu^{-1}(a)$ is preserved by the action of K)

Example: ~~$U(n+1) \curvearrowright \mathbb{P}^n_{\mathbb{C}}$ $(U(n+1) \curvearrowright \mathbb{C}^{n+1})$. $\mathbb{P}^n_{\mathbb{C}}$ is symplectic with the Fubini-Study metric~~

Let $K \simeq S^1 \simeq U(1)$. $K \curvearrowright (\mathbb{P}^n_{\mathbb{C}}, \omega_{fs})$ by $s \cdot [z_0 : \cdots : z_n] = [s^{-1} z_0 : s z_1 : \cdots : s z_n]$.

So S^1 acts via a representation $S^1 \xrightarrow{\rho} U(n+1)$ the moment map of this action is a composition

$\mathbb{P}^n_{\mathbb{C}} \xrightarrow{\mu_{U(n+1)}} \mathfrak{u}(n+1)^* \xrightarrow{\rho^*} \mathfrak{u}(1)^* \simeq \mathbb{R}$. Explicitly: $\mu[z_0 : \cdots : z_n] = \dfrac{-|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2}{\sum_{j=0}^{n} |z_j|^2}$.

Then $\mu^{-1}(0) = \{ [z_0 : z_1 : \cdots : z_n] \mid |z_0|^2 = \sum_{j=1}^{n} |z_j|^2 \}$, so if we identify $(\mathbb{P}^n_{\mathbb{C}})_{x_0 \neq 0} \simeq \mathbb{C}^n$,

then $\mu^{-1}(0) \simeq \{ (z_0, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=0}^{n} |z_j|^2 = 1 \} \simeq S^{2n-1}$ and

$\mu^{-1}(0)/K \simeq S^{2n-1}/S^1 \simeq \mathbb{P}^{n-1}_{\mathbb{C}}$.

(Kempf - Ness) There is an inclusion $\mu^{-1}(0) \subseteq X^{ss}$ which induces a homeomorphism

$$\mu^{-1}(0)/K \simeq X/\!/G.$$

(G complex reductive, K is maximal compact torus)

Some pairs: (G, K); $((\mathbb{C}^\times)^n, (S^1)^n)$, $(GL_n(\mathbb{C}), U(n))$, $(SL_n(\mathbb{C}), SU(n))$.

(Marsden - Weinstein - Meyer) $\mu^{-1}(a)/K$ is a symplectic manifold of dimension $\dim X - 2\dim K$.

In fact, there exists a unique symplectic form ω^{red} such that $\pi^* \omega^{\mathrm{red}} = i^* \omega$

where $\mu^{-1}(a) \xhookrightarrow{i} X$, $\mu^{-1}(a) \xrightarrow{\pi} \mu^{-1}(a)/K$.

- $U(n+1) \curvearrowright (\mathbb{P}^n_{\mathbb{C}}, \omega_{FS})$. ω_{FS} constructing from standard Hermitian inner product which is $U(n+1)$-invariant by definition, so action preserves ω_{FS}. Let $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$.

There is a natural moment map $\mu(\underbrace{[z_0 : \dots : z_n]}_{= [z]}) \cdot \varphi = \dfrac{tr(\bar{z} \varphi z)}{2i \|z\|^2}$ where $[z] \in \mathbb{P}^n_{\mathbb{C}}$, $\varphi \in \mathfrak{u}(n+1)$.

- Let K be a Lie group acting on (X, ω) with $\mu : X \longrightarrow \mathfrak{k}^*$. If $\varphi \in \mathfrak{k}^*$ is fixed by the coadjoint action and the action of K on $\mu^{-1}(\varphi)$ is proper & free then

i) $\mu^{-1}(\varphi)/K$ is a smooth manifold of dimension $\dim X - 2 \dim K$.

ii) There is a unique ω^{red} on $\mu^{-1}(\varphi)/K$ such that $\pi^* \omega^{red} = i^* \omega$ where $\mu^{-1}(\varphi) \overset{i}{\hookrightarrow} X$ is the inclusion and $\pi : \mu^{-1}(\varphi) \longrightarrow \mu^{-1}(\varphi)/K$ is the quotient map.

- Consider $K \simeq S^1 \simeq U(1) \curvearrowright (\mathbb{P}^n_{\mathbb{C}}, \omega_{FS})$ by $s[z_0 : \dots : z_n] = [s^{-1} z_0 : s z_1 : \dots : s z_n]$.

So S^1 acts by a representation $\rho : S^1 \longrightarrow U(n+1)$, the moment map for this S^1 action $\mu : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \text{Lie}(S^1)^* \simeq \mathbb{R}$ is the composition of the moment map $\mu_{U(n+1)} : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathfrak{u}(n+1)^*$ for the $U(n+1)$ action followed by $\rho^* : \mathfrak{u}(n+1)^* \longrightarrow \mathfrak{u}(1)^*$.

$$\mu[z_0 : z_1 : \dots : z_n] = \frac{-|z_0|^2 + |z_1|^2 + \dots + |z_n|^2}{\|z\|^2},$$

$\mu^{-1}(0) = \{[z_0 : \dots : z_n] \mid |z_0|^2 = \sum_{j=1}^{n}$

Categorical quotient : $X \in \mathcal{C}$, $G \curvearrowright X$ $\quad X \xrightarrow{\pi} Y$ $\cdot_1$ π is invariant,

$$\begin{tikzcd} G \times X \arrow[r, "\psi"'] \arrow[d, "p_2"'] & X \\ X & \end{tikzcd} \qquad \pi \circ \psi = \pi \circ p_2$$

$\cdot_2$ $\forall X \xrightarrow{f} Z$ satisfies $\cdot_1$ factors through Y :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

For $k = \bar{k}$, G reductive means all representations split into sum of irreducibles.

For $\mathbb{C}$, it is equivalent to complexification of a compact Lie group $K \subseteq G$.

Very ample : $\exists\, s_0, \dots, s_n$ sections of L with no common zeros such that $[s_0, \dots, s_n] : X \longrightarrow \mathbb{P}^n_k$ is a closed embedding.