535A SPRING 2023 PROBLEM SET #5

Problem 1. Lee *second edition* 5-1

Problem 2. Lee *second edition* 5-4

Problem 3. Lee *second edition* 5-6

Problem 4. Lee *second edition* 5-7

Problem 5. Let $G(k,n)$ denote the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$.

(1) Describe a smooth atlas for $G(k,n)$ with finitely many charts. How many charts does it have? Compute the transition maps (you do not need to explicitly verify that these are smooth but you should try to convince yourself). What is the dimension of $G(k,n)$?

(2) Let $\widetilde{G}(k,n)$ denote the set of n-by-n matrices M which satisfy the following three conditions:
   (a) $M^2 = M$
   (b) M is symmetric
   (c) the trace of M is k.

   Show that there is a natural bijection between $\widetilde{G}(k,n)$ and $G(k,n)$. *Hint: given a linear subspace of $\mathbb{R}^n$, consider the associated orthogonal projection map.*

(3) Prove using the regular level set theorem that $\widetilde{G}(k,n)$ is smooth manifold. If this is too difficult you may specialize to the case $(k,n) = (1,3)$ or even $(k,n) = (1,2)$. *Note: in fact $\widetilde{G}(k,n)$ and $G(k,n)$ are diffeomorphic. In particular, this shows that $G(k,n)$ is diffeomorphic to a real affine variety.*