535A SPRING 2023 PROBLEM SET #1

Problem 1 (Lee *first edition* 1-1)**.** Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff. (This space is called the *line with two origins*.)

Problem 2 (Lee *first edition* 1-2)**.** Show that the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second countable.

Problem 3 (Lee *first edition* 1-3)**.** Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones. [Hint: Begin by constructing homeomorphisms from $\mathbb{B}^n$ to itself that are smooth on $\mathbb{B}^n \setminus \{0\}$.]

Problem 4 (Lee *first edition* 1-5)**.** Let $N = (0, \ldots, 0, 1)$ be the "north pole" in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the "south pole". Define *stereographic projection* $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$, identified with $\mathbb{R}^n$ in the obvious way. Similarly, show that $\widetilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\widetilde{\sigma}$ is called *stereographic projection from the south pole*.)

(b) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1}{|u|^2 + 1}.$$

(c) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\widetilde{\sigma}$ are called *stereographic coordinates*.)

(d) Show that this smooth structure is the same as the one defined in Example 1.20.