Midterm exam
Math 535a: Differential Geometry

1. (20 points) Let M be a smooth manifold which is compact. Prove that there is no smooth submersion from M to $\mathbb{R}^n$ for any $n \geq 1$. *Hint: show that such a submersion would necessarily be an open map, and recall that $\mathbb{R}^n$ is connected.*

Solution: Suppose by contradiction that $F : M \to \mathbb{R}^n$ is a smooth submersion, and put $m = \dim(M)$. Note that $\operatorname{im}(F) \subset \mathbb{R}^n$ is compact, since M is compact and F is continuous, and continuous maps send compact subsets to compact subsets. In particular, $\operatorname{im}(F) \subset \mathbb{R}^n$ is closed, since $\mathbb{R}^n$ is Hausdorff and compact subsets of Hausdorff spaces are closed.

We claim that $\operatorname{im}(F) \subset \mathbb{R}^n$ is also open, meaning that $\operatorname{im}(F) \subset \mathbb{R}^n$ is both closed and open. Since $\mathbb{R}^n$ is connected, the only nonempty closed and open subset is $\mathbb{R}^n$ itself, but F cannot be surjective since $\operatorname{im}(F)$ is compact (whereas $\mathbb{R}^n$ for $n \geq 1$ is not compact), so this gives the desired contradiction.

To justify the above claim, we can invoke the Rank Theorem. Given $q \in \operatorname{im}(F)$, we must show that there exists an open neighborhood of q in $\mathbb{R}^n$ which is contained in $\operatorname{im}(F)$. Given any fixed $p \in F^{-1}(\{q\})$, we can find a smooth chart (U, ϕ) for M centered at p and a smooth chart (V, ψ) for $\mathbb{R}^n$ centered at q such that $F(U) \subset V$ and we have

$$\psi \circ F \circ \phi^{-1}|_{\phi(U)}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

(note that $m \geq n$ since F is a submersion). Since $\phi(U) \subset \mathbb{R}^m$ is an open neighborhood of $\phi(p) = 0$, we can find open neighborhoods A, B of 0 in $\mathbb{R}^n, \mathbb{R}^{m-n}$ respectively such that the Cartesian product $A \times B \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ is an open neighborhood of 0 which is contained in $\phi(U)$. Then $\operatorname{im}(\psi \circ F \circ \phi^{-1}|_{A\times B}) = A$, and therefore we have $\psi^{-1}(A) \subset \operatorname{im}(F)$. Since $\psi^{-1}(A) \subset \mathbb{R}^n$ is an open neighborhood of q contained in $\operatorname{im}(F)$, this justifies the claim.

2. (20 points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map for some $n \geq 1$. Let

$$\operatorname{Gr}(F) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid b = F(a)\} \subset \mathbb{R}^{2n}$$

denote its graph, and let

$$\Delta := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid a = b\} \subset \mathbb{R}^{2n}$$

denote the diagonal. Under what conditions on F do $\operatorname{Gr}(F)$ and Δ intersect transversely as submanifolds of $\mathbb{R}^{2n}$?

Solution: We will give the answer for any smooth map $F : M \to M$ where M is a smooth manifold. Firstly, a point $(a, b) \in M \times M$ lies in $\operatorname{Gr}(F) \cap \Delta$ if and only if $b = F(a) = a$. In other words, the intersection points are in bijective correspondence with the fixed points of F.

The tangent space $T_{(a,b)}(M \times M)$ is naturally isomorphic to $T_aM \times T_bM$, and the tangent spaces $T_{(a,b)}\operatorname{Gr}(F)$ and $T_{(a,b)}\Delta$ are naturally viewed as subspaces of $T_{(a,b)}(M \times M)$. To identify these, note that Δ is the image of the smooth embedding $G : M \to M \times M$ defined by $G(p) = (p, p)$, hence $dG_p(v) = (v, v)$, so we have

$$T_{(a,a)}\Delta = \{(v, w) \in T_aM \times T_aM \mid v = w\}.$$

Similarly, $\operatorname{Gr}(F)$ is the image of the smooth embedding $H : M \to M \times M$ defined by $H(p) = (p, F(p))$, hence $dH_p(v) = (v, dF_p(v))$, so we have

$$T_{(a,F(a))}\operatorname{Gr}(F) = \{(v, w) \in T_aM \times T_{F(a)}M \mid w = dF_a(v)\}.$$

An intersection point $(a,b) \in \mathrm{Gr}(F) \cap \Delta$ is transverse if and only if we have

$$T_{(a,b)}\Delta \cap T_{(a,b)}\mathrm{Gr}(F) = \{0\} \subset T_aM \times T_bM.$$

This fails if and only if there is some $v \in T_aM$ such that $v = dF_a v$, i.e. dF_a has 1 as an eigenvalue.

In summary, $\mathrm{Gr}(F)$ and Δ intersect transversely if and only if for each fixed point p of F, $dF_p : T_pM \to T_pM$ does not have 1 as an eigenvalue.

Aside: a fixed point whose differential does not have 1 *as an eigenvalue is called* nondegenerate*, and these play a fundamental role in fixed point theory and dynamics. A typical goal is to give an a priori lower bound on the number of fixed points of a map $F : M \to M$. The case of all fixed points being nondegenerate is usually the best case scenario for obtaining such a lower bound, whereas degenerate fixed points sometimes ought to count as multiple fixed points, in the same way that the root of $f(x) = x^3$ should really count as three roots.*

3. (15 points) Prove that $\{x^3 - y^3 + xyz - xy = 1\}$ is a smooth submanifold of $\mathbb{R}^3$. Describe the tangent space at the point $(1, 0, 2)$.

Solution: We appeal to the Regular Level Set Theorem. Put $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = x^3 - y^3 + xyz - xy$. Then we have

$$DF_{(x,y,z)} = \begin{pmatrix} 3x^2 + yz - y & -3y^2 + xz - x & xy \end{pmatrix}.$$

The point (x, y, z) is a critical point if and only if this matrix vanishes, which corresponds to the system

$$\begin{cases} 3x^2 + yz - y &= 0 \\ -3y^2 + xz - x &= 0 \\ xy &= 0. \end{cases}$$

For any solution it is easy to see that we must have $x = y = 0$, and then z is unconstrained. We have $F(0, 0, z) = 0$, so 0 is a critical value and all other values are regular values. In particular, 1 is a regular value, so $S := F^{-1}(1)$ is a smoothly embedded submanifold of $\mathbb{R}^3$.

Put $p := (1, 0, 2)$. We describe the tangent space T_pS as a subspace of $T_p\mathbb{R}^3 = \mathbb{R}\langle \frac{\partial}{\partial x}\big|_p, \frac{\partial}{\partial y}\Big|_p, \frac{\partial}{\partial z}\big|_p \rangle$. Recall that we have $T_pS = \ker(dF_p)$. Specializing the above computation to p, we have

$$DF_p = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix},$$

and hence

$$T_pS = \{a \tfrac{\partial}{\partial x}\big|_p + b \tfrac{\partial}{\partial y}\Big|_p + c \tfrac{\partial}{\partial z}\big|_p \mid a, b, c \in \mathbb{R},\ 3a + b = 0\}.$$

4. (20 points) Prove that the map $F : \mathbb{RP}^2 \to \mathbb{RP}^5$ given in projective coordinates by

$$F([x : y : z]) = [x^2 : y^2 : z^2 : yz : xz : xy]$$

is a smooth embedding.

Solution: Since $\mathbb{RP}^2$ is compact, it suffices to show that F is a smooth injective immersion, since then it is automatically a smooth embedding. Let us first check that F is an injective map. Suppose that we have

$$[x_1^2 : y_1^2 : z_1^2 : y_1z_1 : x_1z_1 : x_1y_1] = [x_2^2 : y_2^2 : z_2^2 : y_2z_2 : x_2z_2 : x_2y_2],$$

i.e.

$$(x_1^2, y_1^2, z_1^2 : y_1z_1, x_1z_1, x_1y_1) = \lambda \cdot (x_2^2, y_2^2, z_2^2 : y_2z_2, x_2z_2, x_2y_2)$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. Evidently we must have $\lambda > 0$, and hence we have $x_1 = ax_2\sqrt{\lambda}$, $y_1 = by_2\sqrt{\lambda}$, and $z_1 = cz_2\sqrt{\lambda}$ for some $a, b, c \in \{1, -1\}$.

Suppose first that $x_1 = y_1 = 0$. Then we have $x_2 = y_2 = 0$ and hence $[x_1 : y_1 : z_1] = [0 : 0 : 1] = [x_2 : y_2 : z_2]$. Similarly, if $x_1 = z_1 = 0$ or $y_1 = z_1 = 0$, then we must have $[x_1 : y_1 : z_1] = [x_2 : y_2 : z_2]$.

Now suppose that $x_1 = 0$ but y_1 and z_1 are nonzero. Then y_2 and z_2 are also nonzero and from $y_1/y_2 = \lambda z_2/z_1$ we see that $b = c$, and hence

$$[x_1 : y_1 : z_1] = [0 : y_1 : z_1] = [0 : by_2\sqrt{\lambda} : bz_2\sqrt{\lambda}] = [0 : y_2 : z_2].$$

Similarly, the same is true if we assume y_1 is zero but x_1, z_2 are nonzero, or if z_1 is zero but x_1, y_1 are nonzero.

Finally, suppose that x_1, y_1, z_1 are nonzero. Then the same is true for x_2, y_2, z_2, and we have that x_1/x_2, y_1/y_2, and z_1/z_2 all have the same sign, i.e. $a = b = c$, and hence we have

$$[x_1 : y_1 : z_1] = [ax_2\sqrt{\lambda} : ay_2\sqrt{\lambda} : az_2\sqrt{\lambda}] = [x_2 : y_2 : z_2].$$

To see that F is a smooth immersion, let (U_i, ϕ_i), $i = 1, 2, 3$ denote the standard coordinate charts on $\mathbb{RP}^2$, where U_i consists of all points $[x^1 : x^2 : x^3]$ such that $x^i \neq 0$ and $\phi_i : U_i \to \mathbb{R}^n$ sends $[x^1 : x^2 : x^3]$ to the result after omitting the *ith* coordinate and dividing the remaining coordinates by x^i. Similarly, let (V_i, ψ_i), $i = 1, \ldots, 6$ denote the analogous standard coordinate charts on $\mathbb{RP}^5$.

Observe that $F(U_1) \subset V_1$, and we have $\psi_1 \circ F \circ \phi_1^{-1}(a, b) = \psi_1([1 : a^2 : b^2 : ab : b : a]) = (a^2, b^2, ab, b, a)$. This is a polynomial function, hence smooth, and its Jacobian is the transpose of

$$\begin{pmatrix} 2a & 0 & b & 0 & 1 \\ 0 & 2b & a & 1 & 0 \end{pmatrix},$$

which has rank two for any $a, b \in \mathbb{R}$, hence $F|_{U_1}$ is an immersion. Note that if we permute the components of $[x : y : z]$, then $F([x : y : z])$ changes only by permuting its components, i.e. by postcomposing by a diffeomorphism of $\mathbb{RP}^5$. Since the latter does not affect whether a map is an immersion, this shows that $F|_{U_2}$ and $F|_{U_3}$ are also immersions, and hence F itself is an immersion.

Note: this map F is called the Veronese embedding. *It can be generalized to an embedding of any $\mathbb{RP}^m$ into $\mathbb{RP}^n$ for some n, where the components are all possible monomials of a given degree d.*

5. (25 points) Let M be a smoothly embedded m-dimensional submanifold of $\mathbb{R}^n$ for some $1 \leq m < n$. For each $d < n - m$, prove that there exists a d-dimensional affine subspace of $\mathbb{R}^n$ which is disjoint from M. Is the same true if $d = n - m$?

Solution: We consider affine subspaces S_c of $\mathbb{R}^n$ of the form $\mathbb{R}^d \times \{c\}$ for $c \in \mathbb{R}^{n-d}$. Then S_c intersects M if and only if c lies in the image of the projection map $\pi : M \to \mathbb{R}^{n-d}$ onto the last $n-d$ components. By assumption $n - d > m$, and hence every point in M is a critical point of π, and hence by Sard's theorem the image of π has measure zero. In particular, π is not surjective, so we can indeed find $c \in \mathbb{R}^{n-d} \setminus \operatorname{im}(\pi)$.

In the case $d = n - m$, this argument does not go through. For example in the case $n = 2$ and $d = m = 1$, consider the image M of the smooth embedding $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto (e^t \cos(t), e^t \sin(t))$. M is a spiral, and it intersects every affine line in $\mathbb{R}^2$. Indeed, let $L \subset \mathbb{R}^2$ be some affine line, and let $v \in \mathbb{R}^2 \setminus \{0\}$ be a vector pointing in a direction orthogonal to L. We can $t_1 \in \mathbb{R}$ such that $(e^{t_1} \cos(t_1), e^{t_1} \sin(t_1))$ is an arbitrarily large positive multiple of v, and we can also find t_2 such that $(e^{t_2} \cos(t_2), e^{t_2} \sin(t_2))$ is an arbitrarily large negative multiple of v. Using the Intermediate Value Theorem applied to the projection to along L, it follows that M must intersect L.