Midterm exam
Math 535a: Differential Geometry

1. (20 points) Let M be a smooth manifold which is compact. Prove that there is no smooth submersion from M to $\mathbb{R}^n$ for any $n \geq 1$. *Hint: show that such a submersion would necessarily be an open map, and recall that $\mathbb{R}^n$ is connected.*

2. (20 points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map for some $n \geq 1$. Let

$$\mathrm{Gr}(F) := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid b = F(a)\} \subset \mathbb{R}^{2n}$$

denote its graph, and let

$$\Delta := \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid a = b\} \subset \mathbb{R}^{2n}$$

denote the diagonal. Under what conditions on F do $\mathrm{Gr}(F)$ and Δ intersect transversely as submanifolds of $\mathbb{R}^{2n}$?

3. (15 points) Prove that $\{x^3 - y^3 + xyz - xy = 1\}$ is a smooth submanifold of $\mathbb{R}^3$. Describe the tangent space at the point $(1, 0, 2)$.

4. (20 points) Prove that the map $F : \mathbb{RP}^2 \to \mathbb{RP}^5$ given in projective coordinates by

$$F([x : y : z]) = [x^2 : y^2 : z^2 : yz : xz : xy]$$

is a smooth embedding.

5. (25 points) Let M be a smoothly embedded m-dimensional submanifold of $\mathbb{R}^n$ for some $1 \leq m < n$. For each $d < n - m$, prove that there exists a d-dimensional affine subspace of $\mathbb{R}^n$ which is disjoint from M. Is the same true if $d = n - m$?