Final exam

Math 535a: Differential Geometry

University of Southern California Spring 2021

Instructor: Kyler Siegel

Instructions:

- You are allowed to use our main textbook *Introduction to Smooth Manifolds* by John Lee as much as you wish, as well as the class notes, but you must not consult any other textbook and you must not consult the internet or communicate with other people about any material related to this exam.
- You are welcome to type up with solutions in LaTeX, or write them by hand. Either way you should strive to make your answers are clear, comprehensive, and legible as possible.
- If you have any pressing questions about the wording of a problem, you may email Kyler. He obviously won't be able to help with the actual content of any problem.
- At the top of your exam, please write your name, student id, and the following sentence: "I have adhered to all of the above rules.", followed by your signature.
- Good luck!!

Question:	1	2	3	4	5	Total
Points:	15	15	15	15	15	75
Score:						

1. (15 points) Show that the two-form

$$\frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \{0\}$ is closed but not exact. *Hint: compute its integral over the unit two-sphere.*

2. (I) (5 points) Consider the homogeneous two variable polynomial

$$P(z_1, z_2) = z_1^n + a_{n-1}z_1^{n-1}z_2 + \cdots + a_1 z_1 z_2^{n-1} + a_0 z_2^n$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Let $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ denote the smooth map given by

$$F([z_1 : z_2]) = [P(z_1, z_2) : z_2^n].$$

For each regular point $p \in \mathbb{CP}^1$ of F, show that $dF_p : T_p\mathbb{CP}^1 \to T_{F(p)}\mathbb{CP}^1$ is orientation preserving. Use this to compute the degree of F (in the sense of the last lecture).

(II) (5 points) Let ω be a two-form on $\mathbb{CP}^1$. Show that we have

$$\int_{\mathbb{CP}^1} F^*\omega = n \int_{\mathbb{CP}^1} \omega.$$

(III) (5 points) It turns out that the de Rham cohomology ring of $\mathbb{CP}^2$ is given by a truncated polynomial ring: $H^*(\mathbb{CP}^2) \cong \mathbb{R}[x]/x^3$, with $|x| = 2$. Assuming this, prove that there is no orientation reversing diffeomorphism from $\mathbb{CP}^2$ to itself. *Hint: what would be the induced map $H^4(\mathbb{CP}^2) \to H^4(\mathbb{CP}^2)$?*

3. (15 points) Let $D \subset \mathbb{R}^n$ be a compact subset which is the closure of an open subset with smooth boundary. Prove the following identity for any $f, g \in C^\infty(D)$:

$$\int_D (g\Delta f - f\Delta g)\, dV = \int_{\partial D} (gN(f) - fN(g))\, dS.$$

The notation is as follows:

- $\Delta f \in C^\infty(D)$ denotes the Laplacian $\sum_{i=1}^n \partial_i^2 f$, and similarly for g
- $N \in \Gamma(TD|_{\partial D})$ denotes the unit outward normal vector field along the boundary of D
- $N(f) \in C^\infty(\partial D)$ denotes the directional derivative of f in the direction of N, and similarly for $N(g)$
- $dV = dx^1 \wedge \cdots \wedge dx^n$ denotes the standard volume form on $\mathbb{R}^n$
- $dS \in \Omega^{n-1}(\partial D)$ denotes the induced volume form on ∂D, given by $(dS)_p(v_1, \ldots, v_{n-1}) = (dV)_p(N_p, v_1, \ldots, v_{n-1})$ for any $v_1, \ldots, v_{n-1} \in T_p\partial D$.

Hint: apply Stokes' theorem to the $(n-1)$-form $i_{(g\nabla f - f\nabla g)}dV$.

4. (I) (5 points) Consider the map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $F(x^1, \ldots, x^{n+1}) = (-x^1, \ldots, -x^{n+1})$. Let $I : S^n \to S^n$ be antipodal map, namely the restriction of F to the unit sphere. Prove that I is orientation preserving if and only if n is odd.

(II) (5 points) Prove that $\mathbb{RP}^n$ is orientable for n odd. *Hint: starting with a volume form ω on S^n, show that $\omega + I^*\omega$ descends to a volume form on $\mathbb{RP}^n$.*

(III) (5 points) Prove that $\mathbb{RP}^n$ is *not* orientable for n even. *Hint: given a volume form on $\mathbb{RP}^n$, show that its pullback to S^n is invariant under I, and therefore that I is orientation preserving.*

5. Let G be a compact Lie group of dimension n, and let $\omega \in \Omega^n(G)$ be a volume form which is left-invariant, i.e. $L_g^*\omega = \omega$ for all $g \in G$, where $L_g : G \to G$ denotes left multiplication.

(I) (5 points) Show that $R_g^*\omega$ is also a left-invariant volume form for each $g \in G$. Here $R_g : G \to G$ denotes right multiplication by g. Conclude that for each $g \in G$ we have $R_g^*\omega = \Delta(g)\omega$ for some $\Delta(g) \in \mathbb{R}_{>0}$.

(II) (5 points) Show that $\Delta : G \to \mathbb{R}_{>0}$ is a Lie group homomorphism. Conclude that we must have $\Delta(g) = 1$ for all $g \in G$, and therefore that ω also right invariant.

(III) (5 points) Now let $i : G \to G$ denote the inversion map, i.e. $i(g) = g^{-1}$. Show that we have $i^*\omega = \pm\omega$. Using this, prove that we have

$$\int_G (f \circ i)\omega = \pm \int_G f\omega$$

for any $f \in C^\infty(G)$. *Hint: show that* $i^*\omega$ *is right-invariant.*