535A SPRING 2021 PROBLEM SET #2

Problem 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & x \in (-1,1) \\ 0 & x \in \mathbb{R} \setminus (-1,1). \end{cases}$$

Show that this function is smooth and has compact support. This is called a "bump function". Can you find a real analytic function $\mathbb{R} \to \mathbb{R}$ which has compact support and is not identically zero?

Problem 2. Prove that the open unit ball $B^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \ : \ \sum_{i=1}^n (x^i)^2 < 1\}$ is diffeomorphic to $\mathbb{R}^n$, when both are equipped with their standard smooth structures. *Hint: if you get stuck you may consult problem 1.4 in Tu.*

Problem 3. Let M be any (non-empty) n-dimensional smooth manifold, and let $C^\infty(M)$ denote the set of smooth functions from M to $\mathbb{R}$. Prove that $C^\infty(M)$ is naturally a real vector space, and it is infinite dimensional.

Problem 4. Lee *second edition* 1-9

Problem 5. Lee *second edition* 2-3