520 SPRING 2025 PROBLEM SET #9

Problem 1. Recall that a lattice Λ in $\mathbb{C}$ is a subset of the form

$$\Lambda = \langle \alpha, \beta \rangle = \{k\alpha + l\beta \mid k, l \in \mathbb{Z}\}$$

for two nonzero complex numbers $\alpha, \beta \in \mathbb{C}$ which are linearly independent over $\mathbb{R}$. Since Λ is an additive subgroup of $\mathbb{C}$, we can consider the quotient $\mathbb{C}/\Lambda$ in the sense of groups, i.e. for $z, z' \in \mathbb{C}$ we have $z \sim z'$ if and only if $z - z' \in \Lambda$. Explain why $\mathbb{C}/\Lambda$ is naturally a Riemann surface.

Problem 2. Now consider two lattices $\Lambda = \langle \alpha, \beta \rangle$ and $\Lambda' = \langle \alpha', \beta' \rangle$. Prove that the Riemann surfaces $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are biholomorphic if and only if there exists a nonzero complex number $c \in \mathbb{C}$ and a matrix $A \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$A \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} c\alpha \\ c\beta \end{pmatrix}.$$

Hint: first show that such a biholomorphic exists if and only if there is a biholomorphism $\mathbb{C} \to \mathbb{C}$ *which descends to the quotients. Recall what is* $\mathrm{Aut}(\mathbb{C})$.

Problem 3. Now show for any lattice Λ, there is another lattice of the form $\Lambda' = \langle 1, \tau \rangle$, with $\tau \in \mathbb{H}$, such that $\mathbb{C}/\Lambda$ is biholomorphic to $\mathbb{C}/\Lambda'$. We will denote such a $\mathbb{C}/\Lambda'$ by $\mathbb{T}_\tau$.

Problem 4. Given $\tau, \tau' \in \mathbb{H}$, prove that $\mathbb{T}_\tau$ is biholomorphic to $\mathbb{T}_{\tau'}$ if and only if there is some $A \in \mathrm{PSL}(2, \mathbb{Z})$ whose corresponding Möbius transformation sends τ to τ'. Conclude that the set of Riemann surfaces of the form $\mathbb{C}/\Lambda$ up to biholomorphism is naturally identified with $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$.

Problem 5. Given $A \in \mathrm{SL}(2, \mathbb{R})$, when does its action on $\mathbb{H}$ by Möbius transformations have a fixed point in $\mathbb{H}$? Give a criterion in terms of the trace of A.

Problem 6. Prove that the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathbb{H}$ by Möbius transformations is not free. Compute the stabilizer group of a fixed point.