475 SPRING 2025 PROBLEM SET #7

Problem 1. Find the power series for $f(z) = z\cosh(z^2)$ centered at $z = 0$, and compute the radius of convergence.

Problem 2. Find the power series for $f(z) = \dfrac{z}{z^2+4}$ centered at $z = 0$, and compute the radius of convergence.

Problem 3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(x)$ is real for all $x \in \mathbb{R}$. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series representation for $f(z)$ centered at $z = 0$.

(a) Explain why the radius of convergence is necessarily infinite.
(b) Explain why all of the coefficients $a_0, a_1, a_2, \ldots$ are necessarily real numbers.
(c) Conclude that the real-valued function $f(x)$ for $x \in \mathbb{R}$ has a real Taylor series centered at $x = 0$ with infinite radius of convergence.
(d) Given an example of such an $f(z)$ and compute its power series expansion centered at $z = 0$.
(e) Give an example of a real-valued function $f(x)$ which is everywhere differentiable but does *not* have a real Taylor series centered at $x = 0$ with infinite radius of convergence.

Problem 4. Show that for $0 < |z| < 4$ we have

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Problem 5. Find the Laurent series which represents the function $f(z) = z^2 \sin(1/z^2)$ in the domain $0 < |z| < \infty$.

Problem 6. Consider the function $f(z) = \dfrac{1}{z^2(1-z)}$.

(a) Find the Laurent series which represents $f(z)$ in the domain $0 < |z| < 1$.
(b) Find the Laurent series which represents $f(z)$ in the domain $1 < |z| < \infty$.