475 SPRING 2025 PROBLEM SET #6

Problem 1. Let C be the boundary of the square with vertices $(\pm 2, \pm 2)$, oriented counterclockwise. Evaluate the following integrals:

(a) $\displaystyle\int_C \frac{e^{-z}dz}{z - \pi i/2}$

(b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}dz$

(c) $\displaystyle\int_C \frac{zdz}{2z+1}$

(d) $\displaystyle\int_C \frac{\cosh z}{z^4}dz$

(e) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz$, $-2 < x_0 < 2$.

Problem 2. Let C be the circle $\{|z - i| = 2\}$, oriented counterclockwise. Evaluate the following integrals:

(a) $\displaystyle\int_C \frac{1}{z^2+4}dz$

(b) $\displaystyle\int_C \frac{1}{(z^2+4)^2}dz$

Problem 3. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)dz}{z - z_0} = \int_C \frac{f(z)dz}{(z-z_0)^2}.$$

Problem 4. Suppose that f is a *continuous* function defined on a simple closed contour C. Show that the function

$$g(z) := \frac{1}{2\pi i}\int_C \frac{f(s)ds}{s-z}$$

is *analytic* at each point z in the interior of C, and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$$

at each such point.

Problem 5. Let C be the unit circle, oriented counterclockwise. Show that for each $a \in \mathbb{R}$ we have

$$\int_C \frac{e^{az}}{z}dz = 2\pi i.$$

Then write this integral in terms of θ in order to establish

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi.$$

Problem 6. Let f be continuous on a closed bounded region R, and analytic on nonconstant on the interior of R.

(a) If $f(z) \neq 0$ in R, show that $|f(z)|$ has a minimum value in R, and that this occurs on the boundary of R and not in the interior.

(b) Show that, without the assumption $f(z) \neq 0$, $|f(z)|$ can achieve its minimum value at an interior point.

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function whose real part u satisfies $u(z) \leq 100$ for all $z \in \mathbb{C}$. Prove that f must be constant.