475 SPRING 2025 PROBLEM SET #3

Problem 1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function of the form $f(x + iy) = u(x) + iv(y)$ for $u, v : \mathbb{R} \to \mathbb{R}$. Show that f must be a polynomial of degree at most one.

Problem 2.

(a) Find all entire functions $f(z) = u(z) + iv(z)$ with $u(x + iy) = x^2 - y^2$.
(b) Find all entire functions $f(z) = u(z) + iv(z)$ with $u(x + iy) = x^2 + y^2$.

Problem 3. Given holomorphic functions $f, g : \mathbb{C} \to \mathbb{C}$ and a point $z_0 \in \mathbb{C}$ such that $g(z_0) \neq 0$, prove that the function $h(z) := f(z)/g(z)$ is holomorphic at z_0. *Hint: this is basically asking you to prove the quotient rule for holomorphic functions.*

Problem 4.

(a) Verify the following identities between partial derivatives in rectangular coordinates and polar coordinates[1]:

$$u_r = u_x \cos\theta + u_y \sin\theta, \qquad u_\theta = -u_x r \sin\theta + u_y r \cos\theta$$

and

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \qquad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

(b) Show that the usual rectangular Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

hold if and only if the following equations in polar form hold:

$$ru_r = v_\theta, \qquad u_\theta = -rv_r.$$

(c) Show that we can rewrite the expression $f'(z) = u_x + iv_x$ as

$$f'(z) = e^{-i\theta}(u_r + iv_r),$$

or alternatively as

$$f'(z) = \frac{-i}{z}(u_\theta + iv_\theta).$$

(d) For the function $f(z) = 1/z$ defined on $\mathbb{C} \setminus \{0\}$, show that $f'(z) = -1/z^2$.

Problem 5. Let $f(z) = u(r, \theta) + iv(r, \theta)$ be a holomorphic function defined on $\mathbb{C} \setminus \{0\}$. Show that the real part $u(r, \theta)$ in polar coordinate satisfies

$$r^2 u_{rr}(r, \theta) + ru_r(r, \theta) + u_{\theta\theta}(r, \theta) = 0,$$

and the same folds for $v(r, \theta)$. This is the polar form of Laplace's equation.

[1]Note that u_x is a shorthand for $\frac{\partial}{\partial x} u$ and so on.