475 SPRING 2025 PROBLEM SET #2

Problem 1. Determine the accumulation points of each of the following sets:

(a) $\{(-i)^n \mid n = 1, 2, 3, \dots\}$
(b) $\{3i^n/n \mid n = 1, 2, 3, \dots\}$
(c) $\{2z \neq 0 \mid 0 \leq \arg(z) \leq \pi/2\}$
(d) $\{(-1)^n(1+i)\frac{2n-1}{n} \mid n = 1, 2, 3, \dots\}$

Problem 2. Compute the limit of $f(z) := \left(\frac{z}{\overline{z}}\right)^2$ as $z \to 0$, or show that it does not exist.

Problem 3. Determine the limit $\lim_{z\to\infty} T(z)$, where $T(z) := \dfrac{az+b}{cz+d}$ for some constants $a, b, c, d \in \mathbb{C}$. Determine the limit $\lim_{z\to z_0} T(z)$ for $z_0 \in \mathbb{C}$. Note that your answer may depend on the values of a, b, c, d and z_0.

Problem 4. Using the Cauchy–Riemann equations, show that the following functions are not holomorphic (i.e. complex differentiable):

(a) $f(z) = \overline{z}$
(b) $f(z) = z - \overline{z}$
(c) $f(z) = 2x + ixy^2$
(d) $f(z) = e^x e^{-iy}$.

Problem 5. For the following functions, compute $f'(z)$ or show that it does not exist.

(a) $f(z) = 1/z$
(b) $f(z) = x^2 + iy^2$
(c) $f(z) = x^3 + iy^3$
(d) $f(z) = z\mathrm{Im}(z)$
(e) $f(z) = \overline{z}^2$

Problem 6. Consider the funtion defined by

$$f(x) = \begin{cases} \overline{z}^2/z & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Let u and v denote real and imaginary parts of f. Show that these satisfy the Cauchy–Riemann equations, but nevertheless $f(z)$ is not holomorphic at $z = 0$.