# 475 SPRING 2025 PROBLEM SET #11

**Problem 1.**

(a) Consider the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x, y) = \sin(x)\cosh(y)$. Show that $u$ is harmonic. Find an analytic function $f : \mathbb{C} \to \mathbb{C}$ whose real part is $u$.

(b) Consider the function $u : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ given by $u(x, y) = \dfrac{x}{x^2 + y^2}$. Show that $u$ is harmonic. Find an analytic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ whose real part is $u$.

**Problem 2.** Find the solution to Laplace's equation on the unit disc $\{|z| = 1\}$ with Dirichlet boundary condition on the unit circle given by $\phi(e^{i\theta}) = \sin(3\theta)$.

**Problem 3.**

(a) Let $P(x, y)$ be a real polynomial in two variables having degree two, i.e. $P(x, y) = ax^2 + by^2 + cxy + dx + ey + f$ for some $a, b, c, d, e, f \in \mathbb{R}$. When is $P : \mathbb{R}^2 \to \mathbb{R}$ harmonic?

(b) Now answer the same question but for real polynomials $P(x, y)$ of degree three.

**Problem 4.** Find the image of the infinite strip $\{z = x + iy \mid 0 < y < 5\}$ under the transformation $F(z) = 1/z$. What is it geometrically?

**Problem 5.** Prove that the upper half space $\mathbb{H} := \{z = x + iy \mid y > 0\}$ is biholomorphic to the infinite strip $\{z = x + iy \mid 0 < y < 1\}$. *Hint: think about* $\log(z)$.

**Problem 6.** Prove that the function $F(z) = \frac{1+z}{1-z}$ defines a biholomorphism between the half disc

$$\{z = x + iy \mid x^2 + y^2 < 1, y > 0\}$$

and the first quadrant

$$\{z = x + iy \mid x > 0, y > 0\}.$$

**Problem 7.** Let $\mathrm{SL}(2, \mathbb{R})$ denote the set of 2-by-2 matrices with real entries and determinant equal to 1. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, we consider the corresponding linear fractional transformation[1] $T_A(z) = \dfrac{az + b}{cz + d}$. Verify the following features of linear fractional transformations:

(a) $T$ maps the upper half space $\mathbb{H}$ to itself, i.e. $T : \mathbb{H} \to \mathbb{H}$

(b) Given $A, B \in \mathrm{SL}(2, \mathbb{R})$, we have $T_A \circ T_B = T_{AB}$, where $AB$ is the matrix product of $A$ and $B$.

(c) Given $A \in \mathrm{SL}(2, \mathbb{R})$, $T_A$ is invertible, with inverse given by $T_{A^{-1}}(z)$. *Hint: show that* $T_{\mathbb{1}}(z) = z$, *where* $\mathbb{1}$ *is the identity matrix, and use the previous part.*

(d) The maps $F(z) = z + 1$ and $F(z) = -1/z$ are linear fractional transformations for certain choices of matrices in $\mathrm{SL}(2, \mathbb{R})$.

[1] These are also known as Möbius transformations.

(e) The linear fractional transformation $T_A : \mathbb{H} \to \mathbb{H}$ has a fixed point, i.e. $z_0 \in \mathbb{H}$ such that $T_A(z_0) = z_0$, if and only if $-2 < \mathrm{tr}(A) < 2$, where $\mathrm{tr}(A)$ denotes the trace of the 2-by-2 matrix $A$.