475 SPRING 2025 PROBLEM SET #10

Problem 1. Let C denote the unit circle $\{|z| = 1\}$ with its counterclockwise orientation. For the following functions, compute the contour integral $\int_C \frac{f'(z)}{f(z)} dz$. Then find the zeros and poles of $f(z)$ inside of C and check that these are consistent with the argument principle.

(a) $f(z) = z^2$
(b) $f(z) = 1/z^3$
(c) $f(z) = (2z - 1)^7/z^3$.

Problem 2. Suppose that f is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Assume that f has a unique zero z_0 inside of C with multiplicity m. Show that we have

$$z_0 = \frac{1}{2\pi i m} \int_C \frac{z f'(z)}{f(z)} dz.$$

Problem 3. Determine the number of zeros, counted with multiplicities, of the following functions inside the circle $\{|z| = 1\}$.

(a) $f(z) = z^6 - 5z^4 + z^3 - 2z$
(b) $f(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$
(c) $f(z) = z^7 - 4z^3 + z - 1$.

Problem 4. Determine the number of zeros, counted with multiplicities, of the following functions inside the circle $\{|z| = 2\}$.

(a) $f(z) = z^4 - 2z^3 + 9z^2 + z - 1$
(b) $f(z) = z^5 + 3z^2 + z^2 + 1$.

Problem 5. Determine the number of roots, counted with multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

inside the annulus $\{1 \leq |z| \leq 2\}$.

Problem 6. Show that if c is a complex number such that $|c| > e$, then the equation $cz^n = e^z$ has n roots, counted with multiplicities, inside the circle $\{|z| = 1\}$.