Midterm 1
Math 434: Geometry and transformations
University of Southern California Fall 2022
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Question:	1	2	3	4	Total
Points:	11	10	18	14	53
Score:					

1. (I) (4 points) Let $f : \mathbb{C} \to \mathbb{C}$ be counterclockwise rotation by angle $\theta \in [0, 2\pi)$ about the point $z_0 \in \mathbb{C}$. Write a formula for $f(z)$.

Solution: Let $T(z) = z - z_0$ and $R(z) = e^{i\theta} z$. Then we have

$$f(z) = T^{-1}(R(T(z))) = T^{-1}(e^{i\theta}(z - z_0)) = e^{i\theta}(z - z_0) + z_0.$$

(II) (4 points) Let $g : \mathbb{C} \to \mathbb{C}$ be reflection about the line passing through 0 and $1 + i$. Write a formula for $g(z)$.

Solution: Note that the line makes angle $\pi/4$ with the real axis. Let $R(z) = e^{-i\pi/4} z$. Let $S(z) = \overline{z}$ be the reflection about the real axis. Then we have

$$g(z) = R^{-1}(S(R(z)) = R^{-1}(e^{i\pi/4}\overline{z}) = e^{i\pi/2}\overline{z} = i\overline{z}.$$

(III) (3 points) Let $h : \mathbb{C} \to \mathbb{C}$ be the inversion about the circle with center i and radius 1. What is $h(1+i)$?

Solution: Since $1 + i$ lies on C, we have $h(1 + i) = 1 + i$.

2. (I) (5 points) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x + ay + 2, by + 3)$ for some real constants $a, b \in \mathbb{R}$. For which a, b is this a Euclidean isometry?

Solution: This map is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + B,$$

where

$$A = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

We have seen that this is a Euclidean isometry if and only if A is an orthogonal matrix, i.e. $A^T A = I$, or equivalently the columns of A are orthonormal. Note that (a, b) is orthogonal to $(1, 0)$ if and only if $a = 0$. So we must have $a = 0$ and $b = \pm 1$.

(II) (5 points) Describe all Euclidean isometries $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(0, 0) = (0, 0)$ and $g(1, 0) = (-1, 0)$.

Solution: Any Euclidean isometry fixing $(0, 0)$ is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix},$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an orthogonal 2×2 matrix. Since $g(1, 0) = (-1, 0)$, we must have $a = -1$ and $c = 0$. As in the previous problem, we then have $b = 0$ and $d = \pm 1$. In the first case, $(x, y) \mapsto (-x, y)$ corresponds to reflection about the y-axis. In the second case, $(x, y) \mapsto (-x, -y)$ corresponds to rotation by π about the origin.

3. (I) (4 points) Let $C \subset \mathbb{C}$ be the unit circle centered at the origin, and let $\iota_C : \mathbb{C}_+ \to \mathbb{C}_+$ be its inversion. What is $\iota_C(i/2)$?

Solution: Recall that $\iota_C(z) = \frac{1}{\bar{z}}$. We could derive this by remembering that $\iota_C(re^{i\theta}) = se^{i\theta}$, where s is such that $rs = 1$. So we have

$$\iota_C(re^{i\theta}) = \tfrac{1}{r} e^{i\theta} = \tfrac{1}{\bar{z}}.$$

with $z = re^{i\theta}$.
So $\iota_C(i/2) = (\bar{i}/2)^{-1} = (-i/2)^{-1} = -2/i = 2i$.

(II) (5 points) Write a formula for a hyperbolic transformation $f : \mathbb{D} \to \mathbb{D}$ sending $i/2$ to 0 and $-i$ to 1.

Solution: We can view f as a Möbius transformation $\mathbb{C}_+ \to \mathbb{C}_+$ which maps $\mathbb{D}$ to itself. Since f fixes the unit circle S^1_∞, it must send $\iota_{S^1_\infty}(i/2)$ to $\iota_{S^1_\infty}(0) = \infty$. By the previous part, this means we have $f(2i) = \infty$. So we seek a Möbius transformation $f : \mathbb{C}_+ \to \mathbb{C}_+$ such that $f(i/2) = 0$, $f(2i) = \infty$, and $f(-i) = 1$. This is given by the cross ratio:

$$f(z) = \frac{(z - i/2)(-3i)}{(z - 2i)(-3i/2)} = \frac{2(z - i/2)}{(z - 2i)}.$$

(III) (3 points) Describe the hyperbolic line in $\mathbb{D}^2$ connecting $\frac{1}{3} + \frac{i}{3}$ and $\frac{1}{2} + \frac{i}{2}$.

Solution: Both of these lie on the Euclidean line $\{x + iy \in \mathbb{C} \mid x = y\}$. Since this passes through the origin and hence intersects S^1_∞ in right angles, we find that

$$\{x + iy \in \mathbb{D} \mid x = y\}$$

is the unique hyperbolic line connecting $\frac{1}{3} + \frac{i}{3}$ and $\frac{1}{2} + \frac{i}{2}$.

(IV) (3 points) What is the hyperbolic distance between $\frac{1}{3} + \frac{i}{3}$ and $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$?

Solution: Observe that the second point lies on the circle at infinity, since its modulus is 1. The first point has modulus less thn 1 and hence lies in $\mathbb{D}$. Therefore the distance is infinite.

(V) (3 points) Give an example of a hyperbolic geodesic which is not a Euclidean geodesic.

Solution: Recall that a hyperbolic geodesic is just a hyperbolic line, i.e. a distance minimizing path. Given any circle C which intersects the unit circle S^1_∞ at right angles, $C \cap \mathbb{D}$ is a hyperbolic geodesic. For example, we could consider a circle centered at x with radius r, and try to find conditions which make this perpendicular to S^1_∞.

As a less computational approach, we could simply apply any hyperbolic transformation to the real axis (intersected with $\mathbb{D}$) to get a new hyperbolic line, and then check that it isn't a Euclidean straight line. For example, we could use

$$f(z) = \frac{2(z - i/2)}{z - 2i}$$

from above. Then the image is

$$\left\{\frac{2(x-i/2)}{(x-2i)} \mid x \in (-1,1)\right\}.$$

Note that this is not a Euclidean geodesic, $\frac{2(x-i/2)}{x-2i}$ since is never ∞ for $x \in \mathbb{R} \cup \{\infty\}$.

4. (I) (4 points) Find a formula for a Möbius transformation $f : \mathbb{C}_+ \to \mathbb{C}_+$ such that $f(1) = 0$, $f(2) = 1$, and $f(3) = \infty$.

Solution: This is simply given by a cross ratio:

$$f(z) = \frac{(z-1)(2-3)}{(z-3)(2-1)} = \frac{-(z-1)}{(z-3)} = \frac{-z+1}{z-3}.$$

(II) (3 points) What is $f(\infty)$?

Solution: In general, for the value of $\frac{az+b}{cz+d}$ at ∞ is a/c. In our case we have $f(\infty) = -1$.

(III) (4 points) What is the inverse map $f^{-1} : \mathbb{C}_+ \to \mathbb{C}_+$?

Solution:
The corresponding matrix is

$$\begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix},$$

so its inverse is given by

$$\tfrac{1}{2} \cdot \begin{pmatrix} -3 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -3/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix}$$

Therefore the inverse Möbius transformation is

$$f^{-1}(z) = \frac{-3z/2 - 1/2}{-z/2 - 1/2} = \frac{3z+1}{z+1}.$$

(IV) (3 points) What is the image of the real axis under f?

Solution: Since f sends $1, 2, 3$ to $0, 1, \infty$ respectively, it must send the unique cline joining $1, 2, 3$ to the unique cline joining $0, 1, \infty$. In other words, it sends the real axis to the real axis. (This is also easy to see from the formula, since $\frac{-x+1}{x-3}$ is always real if x is.)