Midterm 2 instructions
Math 425a: Fundamental Concepts of Analysis
University of Southern California Fall 2020
Instructor: Kyler Siegel

Instructions:

- Please write your answers on *blank* (unruled) paper, and make them as legible as possible. You do *not* need to print out the exam. You must make your work and solutions clear for full credit. Please include all scratch work.
- Upload your solutions as a single PDF of sufficiently high resolution. You should use a scanner or camera / smartphone. If you use a smartphone, we recommend using a scanner app. If your writing is not legible we may not be able to give credit (even if it is due only to poor scanning). If you absolutely cannot manage a PDF, please use JPG or other standard image format. Please try to avoid formats such as docx. You may *not* type your solutions, but you are allowed to hand write solutions on a tablet if you prefer.
- Solve as many of the problems as you can in the allotted time, which is *50 minutes*. You will be given an additional 10 minutes as a buffer period to upload your solutions. It is your responsibility to stop taking the exam after 50 minutes and to upload your solutions within the one hour time slot. As a last resort, if you have submission issues you should email your exam as soon as possible to kyler.siegel@usc.edu. Exams not received during the one hour time window might not be accepted.
- I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- Please do not under any circumstances share information about this exam with other students, even after the exam window has ended (in case there are makeup exams). Inquiring about the exam with other students or giving information about the exam to other students is considered a breach of the honor code. Note that this includes even information about the difficulty level of the exam or broad information about what topics are covered. Suspected cases of copying or otherwise cheating will be taken very seriously.
- You may *not* use any electronic devices to complete the exam, apart from those used to view and submit the exam. You are *not* allowed to use any textbook, calculator, pre-written notes, the internet, etc, to aid your solutions. You also may *not* consult with anyone (whether or not they are a student in this course) during the exam. You are expected to follow the honor code.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- You may freely use the restroom during the exam.
- If you have any questions about the exam or think there is a mistake, please interpret the questions as best you soon and give the most reasonable answer you can. Since not everyone will be taking the exam at the same time, it will be difficult to make clarifications in real time.
- At the top of your exam, please write your name, student id, and the following sentence: "I have adhered to all of the above rules.", followed by your signature.
- Good luck!!

Question:	1	2	Total
Points:	72	21	93
Score:			

In the following questions, make your answers as rigorous, comprehensive, and precise as possible. Use your own best judgment as to what is reasonable within the allotted time.

Notation: Recall that $\mathcal{C}$ denotes the standard middle-thirds Cantor set. As usual, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, and $\mathbb{R}$ denotes the real numbers.

1. Determine whether each of the following statements are true or false. If true, give a proof. If false, disprove it by giving a counterexample.

(I) (6 points) Every compact metric space is complete.

> **Solution:** True. Let (p_k) be a Cauchy sequence in a compact metric space M. By sequential compactness, there is a subsequence (p_{i_k}) which converges to a limit $p \in M$. Then the original sequence (p_k) must also converge to p. Indeed, by the Cauchy property, for any $\varepsilon > 0$ we can find $N \in \mathbb{Z}_{\geq 1}$ such that $d(p_k, p_l) < \varepsilon/2$ for $k, l \geq N$. Also, by convergence we can find $M \in \mathbb{Z}_{\geq 1}$ such that $d(p_{i_l}, p) < \varepsilon/2$ for $l \geq M$. Then for $k \geq N$, by the triangle inequality we have $d(p_k, p) \leq d(p_k, p_{i_l}) + d(p_{i_l}, p) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ provided that we take $l \geq \max\{N, M\}$. Since $\varepsilon > 0$ is arbitrary, this shows that every Cauchy sequence converges, hence (M, d) is complete.

(II) (6 points) Every complete metric space is compact.

> **Solution:** False. $\mathbb{R}$ is counterexample.

(III) (6 points) Let $d_k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be the map defined by $d_k(x, y) := |x - y|^k$ for $x, y \in \mathbb{Z}$. Then $(\mathbb{Z}, d_k)$ is a metric space for all positive integers k.

> **Solution:** False. Consider d_2. We have $d_2(1, 3) = |1-3|^2 = 4$. We also have $d_2(1, 2) = |1-2|^2 = 1$ and $d_2(2, 3) = |2 - 3|^2 = 1$. The triangle inequality states that $d_2(1, 3) \leq d_2(1, 2) + d_2(2, 3)$, but this does not hold.

(IV) (6 points) Every path connected metric space is connected.

> **Solution:** True. Suppose by contradiction that (M, d) is a metric space which is path connected but disconnected. Then we can find disjoint nonempty clopen subsets $A, B \subset M$ such that $M = A \cup B$. Pick points $p \in A$ and $q \in B$. By path connectedness, we can find a path $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Then $\gamma^{-1}(A)$ is a nonempty proper clopen subset of $[0, 1]$, contradicting the fact that $[0, 1]$ is connected.

(V) (6 points) Every connected metric space is path connected.

Solution: False. We can take the closed topologist's sine curve $S \subset \mathbb{R}^2$ with the induced metric, given by

$$S = \left\{(x, \sin(1/x)) \in \mathbb{R}^2 \ : \ x \in (0,1]\right\} \cup \left\{(0,y) \in \mathbb{R}^2 \ : \ y \in [-1,1]\right\}.$$

(VI) (6 points) Every compact metric space is bounded.

Solution: True. Suppose (M,d) is compact. Note that M is covered by the collection of all open balls centered at points in M of radius 1. By covering compactness, there is a finite subcovering, i.e. we have $M = B_1(p_1) \cup \cdots \cup B_k(p_k)$ for some $k \in \mathbb{Z}_{\geq 1}$ and points $p_1, \ldots, p_k \in M$. Put

$$R := \max_{1 \leq i,j \leq k} d(p_i, p_j).$$

Then for any two points $x, y \in M$, we have $d(x,y) \leq R + 2$. Indeed, we have $x \in B_1(p_i)$ for some $i \in \{1, \ldots, k\}$ and $y \in B_1(p_j)$ for some $j \in \{1, \ldots, k\}$. Then by the triangle inequality we have

$$d(x,y) \leq d(x,p_i) + d(p_i,p_j) + d(p_j,y) < 1 + R + 1 = R + 2.$$

(VII) (6 points) Every closed and bounded subset of a complete metric space is compact.

Solution: False. Let d denote the discrete metric on $\mathbb{R}$. Then $(\mathbb{R}, d)$ is a complete metric space. Indeed, a sequence in this metric space can only be Cauchy if it is eventually constant, in which case it clearly converges. Observe that $\mathbb{R}$ itself is a closed and bounded subset, but it is not compact. For example, the sequence $1, 2, 3, \ldots$ cannot possibly have any convergent subsequence since the distance with respect to d between any two elements of this sequence is 1.

(VIII) (6 points) Any intersection of countably many open subsets of a given metric space is open.

Solution: False. For example, we have

$$(0,1] = \bigcap_{k=1}^{\infty} (0, 1 + 1/k).$$

Each interval $(0, 1+1/k)$ is open, but $(0,1]$ is not.

(IX) (6 points) Every function $f : [0, 1] \to \mathbb{R}$ is bounded.[1]

> **Solution:** False. For example, we can take $f(1/k) = k$ for each $k \in \mathbb{Z}_{\geq 1}$, and $f(x) = 0$ otherwise. Note that this function is not continuous, and the statement is in fact true for continuous functions by compactness.

(X) (6 points) Every continuous function $f : \mathcal{C} \to \mathbb{R}$ is bounded.

> **Solution:** True. The Cantor set is a closed and bounded subset of $\mathbb{R}$, and hence compact. Since a continuous function maps compact subsets to compact subsets, the image $f(\mathcal{C})$ is compact, hence bounded.

(XI) (6 points) There are at most countably many points of discontinuity for a function $f : \mathbb{R} \to \mathbb{R}$.

> **Solution:** False. If we take $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational, then this function is discontinuous everywhere.

(XII) (6 points) If two metric spaces are homeomorphic and one is complete, then so is the other.

> **Solution:** False. $(0, 1)$ and $\mathbb{R}$ are homeomorphic, e.g. we can take as our homeomorphism the sigmoid function $\sigma : \mathbb{R} \to (0, 1)$ given by $\sigma(x) = e^x/(1 + e^x)$. The latter is complete but the former is not.

2. Consider the following examples of subsets of $\mathbb{R}$:

   (a) $\mathbb{Z}$
   (b) $\mathbb{Q}$
   (c) $\mathbb{R}$
   (d) $\mathbb{R} \setminus \{0\}$
   (e) $\mathcal{C}$

[1] In other words, there is some $M \in \mathbb{R}_{>0}$ such that $|f(x)| \leq M$ for all $x \in [0, 1]$.

We view each of these examples as a metric space with the induced metric from $\mathbb{R}$. For each of the following properties, determine which of the above examples satisfies that property. *Note: there could be several examples (or none) satisfying a given property. You do not need to justify your answer.*

(I) (3 points) compact

Solution: (e)

(II) (3 points) complete

Solution: (a),(c),(e)

(III) (3 points) connected

Solution: (c)

(IV) (3 points) bounded

Solution: (e)

(V) (3 points) open subset of $\mathbb{R}$

Solution: (c),(d)

(VI) (3 points) closed subset of $\mathbb{R}$

Solution: (a),(c),(e)

(VII) (3 points) dense in $\mathbb{R}$

Solution: (b),(c),(d)