Midterm 1 instructions

Math 425a: Fundamental Concepts of Analysis
University of Southern California Fall 2020
Instructor: Kyler Siegel

Instructions:

- Please write your answers on *blank* (unruled) paper, and make them as legible as possible. You do *not* need to print out the exam. You must make your work and solutions clear for full credit. Please include all scratch work.
- Upload your solutions as a single PDF of sufficiently high resolution. You should use a scanner or camera / smartphone. If you use a smartphone, we recommend using a scanner app. If your writing is not legible we may not be able to give credit (even if it is due only to poor scanning). If you absolutely cannot manage a PDF, please use JPG or other standard image format. Please try to avoid formats such as docx. You may *not* type your solutions, but you are allowed to hand write solutions on a tablet if you prefer.
- Solve as many of the problems as you can in the allotted time, which is *50 minutes*. You will be given an additional 10 minutes as a buffer period to upload your solutions. It is your responsibility to stop taking the exam after 50 minutes and to upload your solutions within the one hour time slot. As a last resort, if you have submission issues you should email your exam as soon as possible to kyler.siegel@usc.edu. Exams not received during the one hour time window might not be accepted.
- I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- Please do not under any circumstances share information about this exam with other students, even after the exam window has ended (in case there are makeup exams). Inquiring about the exam with other students or giving information about the exam to other students is considered a breach of the honor code. Note that this includes even information about the difficulty level of the exam or broad information about what topics are covered. Suspected cases of copying or otherwise cheating will be taken very seriously.
- You may *not* use any electronic devices to complete the exam, apart from those used to view and submit the exam. You are *not* allowed to use any textbook, calculator, pre-written notes, the internet, etc, to aid your solutions. You also may *not* consult with anyone (whether or not they are a student in this course) during the exam. You are expected to follow the honor code.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- You may freely use the restroom during the exam.
- If you have any questions about the exam or think there is a mistake, please interpret the questions as best you soon and give the most reasonable answer you can. Since not everyone will be taking the exam at the same time, it will be difficult to make clarifications in real time.
- At the top of your exam, please write your name, student id, and the following sentence: "I have adhered to all of the above rules.", followed by your signature.
- Good luck!!

Question:	1	2	3	4	5	Total
Points:	16	32	14	14	24	100
Score:						

In the following questions, make your answers as rigorous, comprehensive, and explicit as possible. Use your own best judgment as to what is reasonable within the allotted time.

Notation: Recall that $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathcal{P}(S)$ denotes the set of all subsets of a set S.

1. (16 points) Consider the Dedekind cut $x = A|B$, where $A = \{r \in \mathbb{Q} \ : \ r^2 < 5 \quad \text{or} \quad r \leq 0\}$, and $B = \mathbb{Q} \setminus A$. Let y be the Dedekind cut $3^* = C|D$, where $C = \{r \in \mathbb{Q} \ : \ r < 3\}$ and $D = \mathbb{Q} \setminus C$. Describe $x + y$ as a Dedekind cut. *You should be as explicit as possible and try to avoid using irrational numbers.*

Solution: This question is basically asking how to write $3 + \sqrt{5}$ as a Dedekind cut, i.e. of the form $E|F$, where E and F are subsets of the rational numbers satisfying the properties of a Dedekind cut. We should *not* simply write $E = \{r \in \mathbb{Q} \ : \ r < 3 + \sqrt{5}\}$, since the whole point of Dedekind cuts is that they can be described without referencing irrational numbers. However, we may still use irrational numbers to help guide us to the answer. For a rational number r, we have

$$\begin{aligned} r < 3 + \sqrt{5} &\iff r - 3 < \sqrt{5} \\ &\iff (r-3)^2 < 5 \quad \text{or} \quad r - 3 \leq 0. \end{aligned}$$

Therefore we can write $x + y$ as $E|F$, where

$$E = \{r \in \mathbb{Q} \ : \ (r-3)^2 < 5 \quad \text{or} \quad r \leq 3\},$$

and $F = \mathbb{Q} \setminus E$.

2. Describe an example of:

(I) (8 points) a function $\mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective

Solution: We can take the function $f(x) = 2x$.

(II) (8 points) a function $\mathbb{R} \to \mathbb{R}$ which is injective but not surjective.

Solution: We can take $f(x) = \arctan(x)$, or $f(x) = \frac{e^x}{1+e^x}$ (this is called the sigmoid function). It's important that those functions or strictly increasing, otherwise they would not necessarily be injective.

(III) (8 points) a function $\mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective

Solution: Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = \begin{cases} x & x < 0 \\ 0 & x = 0, 1 \\ x - 1 & x \geq 2. \end{cases}$$

(IV) (8 points) an injective map from the set of all functions $\mathbb{N} \to \{0, 1\}$ to $[0, 1]$.

Solution: Given a function $f : \mathbb{N} \to \{0, 1\}$, we send it to the corresponding decimal expansion

$$\frac{f(1)}{10} + \frac{f(2)}{100} + \frac{f(3)}{1000} + \cdots = \sum_{i=1}^{\infty} \frac{f(i)}{10^i} \in [0, 1].$$

This is injective because we never encounter the digit 9, so we don't have to worry about an infinite tail of 9's. Note that it would also suffice to take ternary expansions (i.e. replace 10 with 3 in the above formula).

3. (14 points) Prove the following inequality for any real numbers $x_1, x_2, x_3, y_1, y_2, y_3$:

$$(x_1y_1 + 2x_2y_2 + 3x_3y_3)^2 \leq (x_1^2 + 2x_2^2 + 3x_3^2)(y_1^2 + 2y_2^2 + 3y_3^2).$$

Hint: come up with an inner product. You do not need to prove that it is an inner product.

Solution: Consider the inner product $\langle -, - \rangle$ on $\mathbb{R}^3$ given by

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle := x_1y_1 + 2x_2y_2 + 3x_3y_3.$$

We saw in class that this is indeed an inner product (but you should check that the three axioms hold if you are not convinced!). The Cauchy–Schwarz inequality then directly gives the desired result after squaring both sides.

4. (14 points) Given two real numbers $x, y \in \mathbb{R}$, let us put $x \sim y$ if and only if $|y - x| \leq 1$. Does this define an equivalence relation on $\mathbb{R}$? You should justify your answer.

Solution: No, it is not transitive. For example, we have $13 \sim 14$ and $14 \sim 15$, but we do not have $13 \sim 15$.

5. Determine whether each of the following sets is countable or uncountable. *You must provide some justification for full credit, but it does not need to be a complete proof.*

(I) (8 points) the set of all real numbers x such that x^{100} is a rational number

Solution: This is countable. For each rational number r there are at most 100 real numbers x such that $x^{100} = r$. Therefore this countable union of finite sets, which is countable. *Note: actually for any $r > 0$ there are two real numbers x such that $x^{100} = r$, one positive and one negative. The rest of the 100 roots are complex numbers.*

(II) (8 points) the set of all positive integers which are *not* prime

Solution: This is countable, being the a subset of the countable set $\mathbb{Z}$.

(III) (8 points) the set of all tuples $(x_1, \ldots, x_5) \in \mathbb{R}^5$ such that $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 1$.

Solution: This is uncountable. For example, it contains all tuples of the form $(r, 0, 0, 0, 0)$ with $r \in [0, 1]$, which is clearly in bijection with $[0, 1]$, which is uncountable. Therefore this set contains a subset which is uncountable, hence it must also be uncountable.