Midterm 1

Math 125: Calculus I
University of Southern California Fall 2022
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Question:	1	2	3	4	Total
Points:	24	18	12	12	66
Score:					

1. Evaluate the following limits:

(I) (6 points)

$$\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}$$

Solution: Note that plugging in $x = 4$ gives 0 for both the numerator and denominator, so this is inconclusive. However, we have

$$\frac{x^2-4x}{x^2-3x-4} = \frac{x(x-4)}{(x-4)(x+1)},$$

and hence we have

$$\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{5}.$$

(II) (6 points)

$$\lim_{x\to -2} \frac{2-|x|}{2+x}$$

Solution: Since we are taking the limit $x \to -2$, we only care about x values which are less than 2, and in this regime we have $|x| = -x$. So we have

$$\lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{x\to -2} \frac{2+x}{2+x} = \lim_{x\to -2} 1 = 1.$$

(III) (6 points)

$$\lim_{x \to 0} \frac{\cos(x) - 1}{\sin(x)}$$

Solution: Note that plugging in $x = 0$ gives 0 for the numerator and denominator, so this is inconclusive. However, recall that we have

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

and

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0.$$

The first identity above is a little tricky to derive but we gave a proof in class and we've seen it many times. If you don't remember the second one, recall that it can be derived (from the first one) as follows:

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos(x) - 1}{x} = \lim_{x \to 0} \frac{(\cos(x) - 1)(\cos(x) + 1)}{x(\cos(x) + 1)} &= \lim_{x \to 0} \frac{\cos^2(x) - 1}{x(\cos(x) + 1)} \\
&= \lim_{x \to 0} \frac{-\sin^2(x)}{x(\cos(x) + 1)} \\
&= -\lim_{x \to 0} \frac{\sin(x)}{x} \frac{\sin(x)}{(\cos(x) + 1)} \\
&= -\lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{\sin(x)}{\cos(x) + 1} \\
&= -1 \cdot 0 = 0.
\end{aligned}$$

Back to the task at hand, we have

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos(x) - 1}{\sin(x)} &= \lim_{x \to 0} \frac{\cos(x) - 1}{x} \lim_{x \to 0} \frac{x}{\sin(x)} \\
&= \lim_{x \to 0} \frac{\cos(x) - 1}{x} \left(\lim_{x \to 0} \frac{\sin(x)}{x} \right)^{-1} \\
&= 0 \cdot 1 = 0.
\end{aligned}$$

(IV) (6 points)

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 4x + 1} - x \right)$$

Solution: We have

$$
\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2+4x+1}-x) &= \lim_{x\to\infty}\frac{(\sqrt{x^2+4x+1}-x)(\sqrt{x^2+4x+1}+x)}{\sqrt{x^2+4x+1}+x}\\
&= \lim_{x\to\infty}\frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}\\
&= \lim_{x\to\infty}\frac{4x+1}{\sqrt{x^2+4x+1}+x}\\
&= \lim_{x\to\infty}\frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1}\\
&= \frac{4}{2}=2.
\end{aligned}
$$

2. Compute the derivatives of the following functions:

(I) (6 points)

$$\sin(\cos(x^5))$$

Put $f(x)=\sin(x)$ and $g(x)=\cos(x^5)$. By the chain rule we have

$$\frac{d}{dx}\sin(\cos(x^5)) = f'(g(x))g'(x) = \cos(g(x))g'(x).$$

Using again the chain rule, we have $g'(x)=\frac{d}{dx}\cos(x^5) = -\sin(x^5)\cdot 5x^4$. Therefore we have

$$\frac{d}{dx}\sin(\cos(x^5)) = -\cos(\cos(x^5))\sin(x^5)\cdot 5x^4.$$

(II) (6 points)

$$\frac{x^4-1}{x^4+1}$$

Solution: We use the quotient rule, with $f(x)=x^4-1$ and $g(x)=x^4+1$. So we have

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{x^4-1}{x^4+1}\right) &= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}\\
&= \frac{4x^3(x^4+1)-(x^4-1)(4x^3)}{(x^4+1)^2}\\
&= \frac{4x^7+4x^3-4x^7+4x^3}{(x^4+1)^2}\\
&= \frac{8x^3}{(x^4+1)^2}.
\end{aligned}
$$

(III) (6 points)

$$x^5\sqrt{x^3+1}$$

Solution: We use the product rule with $f(x) = x^5$ and $g(x) = \sqrt{x^3+1}$. Note that by the chain rule we have $g'(x) = \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2$. So we have:

$$\begin{aligned}\frac{d}{dx}(x^5\sqrt{x^3+1}) &= f'(x)g(x) + f(x)g'(x) \\ &= 5x^4\sqrt{x^3+1} + x^5 \cdot \frac{3x^2}{2\sqrt{x^3+1}} \\ &= 5x^4\sqrt{x^3+1} + \frac{3x^7}{2\sqrt{x^3+1}}.\end{aligned}$$

It would be fine to leave this answer as is, or we could clean it up a little bit more as

$$\frac{10x^4(x^3+1)+3x^7}{2\sqrt{x^3+1}} = \frac{13x^7+10x^4}{2\sqrt{x^3+1}}.$$

3. (12 points) Consider the curve $x^2 + 4xy + y^2 = 13$. Find the equation of the tangent line at the point $(2, 1)$. What is the equation of the normal line at the same point?

Solution: Using implicit differentiation, we get

$$2x + 4y + 4x\tfrac{dy}{dx} + 2y\tfrac{dy}{dx} = 0.$$

Plugging in $(2, 1)$ for (x, y), we have

$$4 + 4 + 8\tfrac{dy}{dx} + 2\tfrac{dy}{dx} = 0,$$

and hence that slope of the tangent line at $(2, 1)$ is

$$\tfrac{dy}{dx} = -8/10 = -4/5.$$

Therefore the equation of the tangent line at $(2, 1)$ is

$$(y - 1) = -\tfrac{4}{5}(x - 2).$$

To get the normal line, we simply replace $-4/5$ with its negative reciprocal, namely $\frac{5}{4}$, so we get

$$(y - 1) = \tfrac{5}{4}(x - 2).$$

4. (12 points) Suppose that the volume of a cube is increasing at a rate of 30 cubic meters per second. How fast is the surfacea area increasing at the moment when the length of an edge is 5 meters?

Solution: Let x be the length of an edge of the cube. Then the volume is given by $V(x) = x^3$, and the surface area is given by $S(x) = 6x^2$ (recall that a cube has six faces, each of which is a square). We have

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

and hence

$$\frac{dx}{dt} = \frac{1}{3x^2} \frac{dV}{dt}.$$

We then have

$$\frac{dS}{dt} = 12x \frac{dx}{dt} = \frac{4}{x} \frac{dV}{dt}.$$

We are given $\frac{dV}{dt} = 30$, so at the moment when $x = 5$ we have

$$\frac{dS}{dt} = \frac{4}{5} \cdot 30 = 24,$$

and this is given in square meters per second.