Final exam

Math 535a: Differential Geometry

University of Southern California Spring 2021

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Instructions:

- You are allowed to use our main textbook *Introduction to Smooth Manifolds* by John Lee as much as you wish, as well as the class notes, but you must not consult any other textbook and you must not consult the internet or communicate with other people about any material related to this exam.
- You are welcome to type up with solutions in LaTeX, or write them by hand. Either way you should strive to make your answers are clear, comprehensive, and legible as possible.
- If you have any pressing questions about the wording of a problem, you may email Kyler. He obviously won't be able to help with the actual content of any problem.
- At the top of your exam, please write your name, student id, and the following sentence: "I have adhered to all of the above rules.", followed by your signature.
- Good luck!!

Question:	1	2	3	4	5	Total
Points:	15	15	15	15	15	75
Score:						

1. (15 points) Show that the two-form

$$\frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \{0\}$ is closed but not exact. *Hint: compute its integral over the unit two-sphere.*

Solution: Let α denote the two-form at hand. Putting $r = \sqrt{x^2 + y^2 + z^2}$, we have

$$\partial_x \left(\frac{x}{r^3} \right) = r^{-3} - 3x^2 r^{-5},$$

with similar expressions after replacing x by y or z. We therefore have

$$\begin{aligned} d\alpha &= (r^{-3} - 3x^2 r^{-5}) dx \wedge dy \wedge dz - (r^{-3} - 3y^2 r^{-5}) dy \wedge dx \wedge dz + (r^{-3} - 3z^2 r^{-5}) dz \wedge dx \wedge dy \\ &= (3r^{-3} - 3(x^2 + y^2 + z^2) r^{-5}) dx \wedge dy \wedge dz = 0. \end{aligned}$$

To see that α is not exact, let $\iota : S^2 \to \mathbb{R}^3$ denote the inclusion of the unit sphere. It suffices to show that we have

$$\int_{S^2} \iota^* \alpha \neq 0.$$

Indeed, if we had $\alpha = d\beta$, then Stokes' theorem would give

$$\int_{S^2} \iota^* \alpha = \int_{S^2} \iota^*(d\beta) = \int_{S^2} d\iota^*(\beta) = \int_{\partial S^2} \iota^* \beta = 0.$$

Let compute the integral in spherical coordinates

$$(x, y, z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

where $\theta \in [0, 2\pi)$ measures the angle in the xy-plane and $\phi \in (0, \pi)$ measures the angle from the positive z-axis. Strictly speaking this gives a parametrization of S^2 minus the north and south poles, but that makes no difference for the integral since this is a subset of full measure. We have

$$\begin{aligned}\iota^* dx &= \cos\phi\cos\theta d\phi - \sin\phi\sin\theta d\theta \\ \iota^* dy &= \cos\phi\sin\theta d\phi + \sin\phi\cos\theta d\theta \\ \iota^* dz &= -\sin\phi d\phi\end{aligned}$$

and therefore in (θ, ϕ) coordinates we have (after some simplifications)

$$\iota^*\alpha = -\sin\phi d\theta \wedge d\phi. \tag{1}$$

Now we can convert $\int_{S^2} \iota^*\alpha$ into an ordinary multivariate integral. We have (at least up to a sign depending on the orientation of S^2, which makes no difference for us):

$$\int_{S^2} \iota^*\alpha = \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} \sin\phi d\theta \wedge d\phi = -4\pi. \tag{2}$$

2. (I) (5 points) Consider the homogeneous two variable polynomial

$$P(z_1, z_2) = z_1^n + a_{n-1}z_1^{n-1}z_2 + \cdots + a_1 z_1 z_2^{n-1} + a_0 z_2^n$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Let $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ denote the smooth map given by

$$F([z_1 : z_2]) = [P(z_1, z_2) : z_2^n].$$

For each regular point $p \in \mathbb{CP}^1$ of F, show that $dF_p : T_p\mathbb{CP}^1 \to T_{F(p)}\mathbb{CP}^1$ is orientation preserving. Use this to compute the degree of F (in the sense of the last lecture).

Solution:

Let $U = \{[z : 1]\} \subset \mathbb{CP}^1$ be one of the standard charts for $\mathbb{CP}^1$, with complex coordinate z or alternatively real coordinates x, y, where $z = x + iy$. Note that F maps U to itself, and maps $[0, 1]$ to $[0, 1]$. With respect to the complex coordinate z we have $F(z) = P(z)$. In particular, for any $p = [z : 1] \in U$, dF_p is identified with a complex linear map $\mathbb{C} \to \mathbb{C}$, namely multiplication by $P'(z)$. In particular, as a complex linear map, the real determinant is nonnegative.

Let $q \in U$ be any regular value of P, which we can view also as a regular value of F. For each $p \in P^{-1}(q)$, dF_p is nonzero. By the fundamental theorem of algebra and the fact that q is a regular value, we see that $\{p \in \mathbb{C} \mid P(p) = q\}$ has exactly n solutions. Therefore we have

$$\deg(F) = \sum_{p \in P^{-1}(q)} \deg_p F = n.$$

(II) (5 points) Let ω be a two-form on $\mathbb{CP}^1$. Show that we have

$$\int_{\mathbb{CP}^1} F^*\omega = n \int_{\mathbb{CP}^1} \omega.$$

Solution: Note that $\omega \in \Omega^2(\mathbb{CP}^1)$ is automatically closed. Since $\deg(F) = n$, using the definition of degree in terms of the induced map on top degree de Rham cohomology we have $[F^*\omega] = n[\omega] \in H^2(\mathbb{CP}^1)$. As a corollary to Stokes' theorem, $\int_{\mathbb{CP}^1} F^*\omega$ depends only on the de Rham cohomology class of $F^*\omega$, and hence we have

$$\int_{\mathbb{CP}^1} F^*\omega = \int_{\mathbb{CP}^1} n\omega.$$

(III) (5 points) It turns out that the de Rham cohomology ring of $\mathbb{CP}^2$ is given by a truncated polynomial ring: $H^*(\mathbb{CP}^2) \cong \mathbb{R}[x]/x^3$, with $|x| = 2$. Assuming this, prove that there is no orientation reversing diffeomorphism from $\mathbb{CP}^2$ to itself. *Hint: what would be the induced map $H^4(\mathbb{CP}^2) \to H^4(\mathbb{CP}^2)$?*

Solution: Suppose that $F : \mathbb{CP}^2 \to \mathbb{CP}^2$ is a smooth map. Recall that the pullback F^* induces a (graded) ring homomorphism on de Rham cohomology $F^*H^*(\mathbb{CP}^2) \to H^*(\mathbb{CP}^2)$. If F is moreover a diffeomorphism, then this a ring isomorphism, and we have $F^*(x) = cx$ for some $c \in \mathbb{R} \setminus \{0\}$, and consequently we have

$$F^*(x^2) = F^*(x) \cdot F^*(x) = c^2x^2.$$

It follows that F^* pull back positive volume forms to positive volume forms, i.e. it is orientation preserving. We conclude that there is no orientation reversing diffeomorphism of $\mathbb{CP}^2$. Food for thought: what about $\mathbb{CP}^3$?

3. (15 points) Let $D \subset \mathbb{R}^n$ be a compact subset which is the closure of an open subset with smooth boundary. Prove the following identity for any $f, g \in C^\infty(D)$:

$$\int_D (g\Delta f - f\Delta g)\, dV = \int_{\partial D} (gN(f) - fN(g))\, dS.$$

The notation is as follows:

- $\Delta f \in C^\infty(D)$ denotes the Laplacian $\sum_{i=1}^n \partial_i^2 f$, and similarly for g
- $N \in \Gamma(TD|_{\partial D})$ denotes the unit outward normal vector field along the boundary of D
- $N(f) \in C^\infty(\partial D)$ denotes the directional derivative of f in the direction of N, and similarly for $N(g)$
- $dV = dx^1 \wedge \cdots \wedge dx^n$ denotes the standard volume form on $\mathbb{R}^n$
- $dS \in \Omega^{n-1}(\partial D)$ denotes the induced volume form on ∂D, given by $(dS)_p(v_1, \ldots, v_{n-1}) = (dV)_p(N_p, v_1, \ldots, v_{n-1})$ for any $v_1, \ldots, v_{n-1} \in T_p\partial D$.

Hint: apply Stokes' theorem to the $(n-1)$-form $i_{(g\nabla f - f\nabla g)}dV$.

Solution: Note: this is known as Green's second identity. Following the hint, put

$$\alpha := i_{(g\nabla f - f\nabla g)} dV \in \Omega^{n-1}(D).$$

According to Stokes' theorem, we have

$$\int_{\partial D} \alpha|_{\partial D} = \int_D d\alpha.$$

We can write the left hand side as

$$\int_{\partial D} (g i_{\nabla f} dV - f i_{\nabla g} dV)|_{\partial D}.$$

Along ∂D, we can write $\nabla f = \langle \nabla f, N\rangle N + X$, where X is a vector field on ∂D (i.e. perpendicular to N). Note that we have $(i_X dV)|_{\partial D} = 0$. Indeed, given any input vectors $v_1, \ldots, v_{n-1} \in T_p\partial D$, we have $i_X dV(v_1, \ldots, v_{n-1}) = dV(X, v_1, \ldots, v_{n-1}) = 0$ since the vectors $X, v_1, \ldots, v_n$ are necessarily linearly dependent. Therefore, we have

$$(i_{\nabla f} dV)|_{\partial D} = (i_{\langle \nabla f, N\rangle N} dV)|_{\partial D} = \langle \nabla f, N\rangle i_N dV = N(f) dS.$$

Similarly, we have $(i_{\nabla g} dV)|_{\partial D} = N(g)dS$, and hence

$$\int_{\partial D} \alpha|_{\partial D} = \int_{\partial D} (gN(f) - fN(g)) dS.$$

As for the right hand side, note that we have

$$d\alpha = dg \wedge i_{\nabla f} dV + g d(i_{\nabla f} dV) - df \wedge i_{\nabla g} dV - f d(i_{\nabla g} dV).$$

We have

$$\begin{aligned} dg \wedge i_{\nabla f} dV &= dg \wedge i_{\sum_{i=1}^n (\partial_i f)\partial_i} dV \\ &= \sum_{i=1}^n \partial_i f dg \wedge i_{\partial_i} dV \\ &= \sum_{i=1}^n \partial_i f \partial_i g dx^i \wedge i_{\partial_i} dV \\ &= \sum_{i=1}^n \partial_i f \partial_i g dV. \end{aligned}$$

This is also equal to $df \wedge i_{\nabla g} dV$ by symmetry, so we have

$$dg \wedge i_{\nabla f} dV - df \wedge i_{\nabla g} dV = 0.$$

Meanwhile, we claim that $d(i_{\nabla f} dV) = \Delta f dV$. Indeed, we have

$$\begin{aligned} d(i_{\nabla f} dV) &= d\Big(\sum_{i=1}^n \partial_i f i_{\partial_i} dV\Big) && (3) \\ &= \sum_{i=1}^n d(\partial_i f) i_{\partial_i} dV && (4) \\ &= \sum_{i=1}^n \partial_i^2 f dx^i \wedge i_{\partial_i} dV && (5) \\ &= \sum_{i=1}^n \partial_i^2 f dV && (6) \\ &= (\Delta f) dV. && (7) \end{aligned}$$

By symmetry, we also have $d(i_{\nabla g} dV) = (\Delta g) dV$. We therefore have $\int_D d\alpha = \int_D (g\Delta f - g\Delta f) dV$, as desired.

4. (I) (5 points) Consider the map $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by $F(x^1, \ldots, x^{n+1}) = (-x^1, \ldots, -x^{n+1})$. Let $I : S^n \to S^n$ be antipodal map, namely the restriction of F to the unit sphere. Prove that I is orientation preserving if and only if n is odd.

Solution: Let $p_N = (0, \ldots, 0, 1)$ and $p_S = I(p_N) = (0, \ldots, 0, -1)$. It suffices to check that $dI_{p_N} : T_{p_N} S^n \to T_{p_S} S^n$ is orientation preserving if and only if n is odd. Let $U_\pm \subset S^n$ denote the subset of points with $\pm x_{n+1} > 0$. We have graph coordinates $(x_1, \ldots, x_n)$ on $U_\pm$, where $x_{n+1} = \pm\sqrt{1 - \sum_{i=1}^n (x^i)^2}$. Using the orientation of the unit sphere as the boundary of the unit ball, we see that one of these charts is positively oriented and the other is negatively oriented, and hence it suffices to show that the corresponding Jacobian matrix DI_{p_N} has positive determinant if and only if n is even. With respect to these coordinates, we have $dI_{p_N}(\partial_{x^i}) = -\partial_{x^i}$ for $i = 1, \ldots n$, and hence the corresponding Jacobian determinant is $(-1)^n$, as required.

(II) (5 points) Prove that $\mathbb{RP}^n$ is orientable for n odd. *Hint: starting with a volume form ω on S^n, show that $\omega + I^*\omega$ descends to a volume form on $\mathbb{RP}^n$.*

Solution: Let ω be any volume form on S^n, and put $\alpha := \omega + I^*\omega$. Then we have $I^*\alpha = I^*\omega + I^*I^*\omega = \alpha$. Moreover, since I is orientation preserving for n odd, ω and $I^*\omega$ are both volume forms inducing the same orientation on S^n, and hence α is also a volume form on S^n. It follows that α descends to a volume form β on $\mathbb{RP}^n$. Indeed, let $\pi : S^n \to \mathbb{RP}^n$ denote the projection map. Given $q \in \mathbb{RP}^n$ and $p \in \pi^{-1}(q)$, and $v_1, \ldots, v_n \in T_q\mathbb{RP}^n$, we put $\beta(v_1, \ldots, v_n) := \alpha(d\pi_p^{-1}(v_1), \ldots, d\pi_p^{-1}(v_n))$. Since α is invariant under I, it is easy to check that this is well-defined and nowhere vanishing.

(III) (5 points) Prove that $\mathbb{RP}^n$ is *not* orientable for n even. *Hint: given a volume form on $\mathbb{RP}^n$, show that its pullback to S^n is invariant under I, and therefore that I is orientation preserving.*

Solution: Suppose by contradiction that β is a volume form on $\mathbb{RP}^n$. Then $\alpha := \pi^*\beta$ is a volume form on S^n satisfying $I^*\alpha = \alpha$. In particular, this shows that I is orientation preserving, which is not the case for n even.

5. Let G be a compact Lie group of dimension n, and let $\omega \in \Omega^n(G)$ be a volume form which is left-invariant, i.e. $L_g^*\omega = \omega$ for all $g \in G$, where $L_g : G \to G$ denotes left multiplication.

(I) (5 points) Show that $R_g^*\omega$ is also a left-invariant volume form for each $g \in G$. Here $R_g : G \to G$ denotes right multiplication by g. Conclude that for each $g \in G$ we have $R_g^*\omega = \Delta(g)\omega$ for some $\Delta(g) \in \mathbb{R}_{>0}$.

Solution: Since left and right multiplication commute, for any $g, h \in G$ we have

$$L_h^*(R_g^*\omega) = (R_g \circ L_h)^*\omega = (L_h \circ R_g)^*\omega = R_g^* L_h^*\omega = R_g^*\omega. \tag{8}$$

Recall that there is a unique left-invariant n-form on G up to scaling. Indeed, for each $g \in G$, ω_g is uniquely determined by ω_e via the relation $\omega_g = L_{g^{-1}}^*\omega_e$, and ω_e can be any chosen element of $\Lambda^n(T_e^*G)$ (c.f. Haar measure). In particular, we have $R_g^*\omega = C\omega$ for some nonzero constant C, which we denote by $\Delta(g)$. Since $\Delta : G \to \mathbb{R}$ is continuous and never zero, and clearly $\Delta(e) = 1$, it follows that $\Delta(g)$ is positive for all $g \in G$.

(II) (5 points) Show that $\Delta : G \to \mathbb{R}_{>0}$ is a Lie group homomorphism. Conclude that we must have $\Delta(g) = 1$ for all $g \in G$, and therefore that ω also right invariant.

Solution: Note that we have $\Delta(g) = \int_G R_g^*\omega / \int_G \omega$, and it follows that Δ is smooth. To see that Δ is a group homomorphism, note that for any $g, h \in G$ we have $\Delta(gh)\omega = R_{gh}^*\omega = (R_h \circ R_g)^*\omega = R_g^* R_h^*\omega = R_g^*(\Delta(h)\omega) = \Delta(h)\Delta(g)\omega$, and hence we have $\Delta(gh) = \Delta(g)\Delta(h)$.
Since $\Delta : G \to \mathbb{R}_{>0}$ is continuous and G is compact, the image must be compact. Since the image is also a subgroup pf $\mathbb{R}_{>0}$, it must be $\{1\}$, and hence we have $\Delta \equiv 1$. Therefore for any $g \in G$ we have $R_g^*\omega = \omega$, i.e. ω is right invariant.

(III) (5 points) Now let $i : G \to G$ denote the inversion map, i.e. $i(g) = g^{-1}$. Show that we have $i^*\omega = \pm\omega$. Using this, prove that we have

$$\int_G (f \circ i)\omega = \pm \int_G f\omega$$

for any $f \in C^\infty(G)$. *Hint: show that $i^*\omega$ is right-invariant.*

Solution: Note that we have $i \circ R_g = L_{g^{-1}} \circ i$. Therefore we have

$$R_g^* i^*\omega = (i \circ R_g)^*\omega = (L_{g^{-1}} \circ i)^*\omega = i^* L_{g^{-1}}^*\omega = i^*\omega. \tag{9}$$

This shows that $i^*\omega$ is right invariant. Since ω is also right invariant, we must have $i^*\omega = C\omega$ for some constant. Then we have

$$\omega = (i \circ i)^*\omega = i^* C\omega = C^2\omega,$$

and hence $C = \pm 1$, whence $i^*\omega = \pm\omega$.
Finally, by the pullback formula for integrals have

$$\int_G f\omega = \pm \int_G (f \circ i) i^*\omega = \int_G (f \circ i)\omega.$$

Note that the signs cancel since $i^*\omega = \omega$ if i is orientation-preserving, otherwise $i^*\omega = -\omega$ if i is orientation-reversing. A simple example to think about is $\mathbb{T}^n$, where $\omega = d\theta_1 \wedge \cdots \wedge d\theta_n$ and the inversion map $i(\theta_1, \ldots, \theta_n) = (-\theta_1, \ldots, -\theta_n)$ is orientation-preserving if and only if n is even.